

The Hopf Fibration as a Working Bundle of Einstein–Cartan–Evans Theory

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Abstract

ECE theory reads Cartan geometry as physics: the tetrad is a potential, torsion is a field strength, and the first Bianchi identity supplies the homogeneous field equation. Currents (ordinary and homogeneous) consist of the geometric fields themselves; when those fields are sufficiently compact they appear macroscopically as charges and currents (Volume 1, Eqs. 4.45–4.46 [2]; UFT 456 [10]). This paper supplies one compact bundle on which those identifications can be written globally, not only in a local frame. The bundle is the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$. It is not a new axiom. It is a choice of total space on which the existing ECE axioms stay intact and several unfinished UFT details acquire a geometric home: the third tetrad component of a circularly polarised wave (UFT 439), the 4π spinor circuit, the integer helicity of a hopfion, and a candidate geometric carrier for the potential flux V of UFT 454–458. A later section records Clifford algebra (geometric algebra) as the bookkeeping language used in UFT 456 (Eckardt and Goel) for the J – V duality; it is not a new axiom. Plain-language remarks follow each technical section and each newly introduced concept.

Keywords: ECE theory, Cartan geometry, tetrad, spin connection, torsion, Hopf fibration, $B^{(3)}$, potential flux V , Beltrami field, Clifford algebra, space-time algebra, UFT 439, UFT 454–458.

How to read this paper. Sections 1–4 set out the ECE axioms and the geometry of the 3-sphere. Sections 5–12 align that geometry with specific UFT papers. Section 13 lists the identities computed with computer algebra; Section 14 applies them to eight open UFT details. Sections 15–22 collect what is not claimed, a dictionary, the status of supporting calculations, proposed tests, conclusions, definitions, symbols and acronyms. Every equation block and figure is followed by a short plain-language note (“In words”, “What the figure shows”, or “Remark”). *Layer 1* marks statements that follow from geometry and the ECE axioms alone; *Layer 2* marks proposed physical readings that still need a calculation or a measurement. Footnotes explain technical terms.

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1 Purpose and limits

The working method of UFT 439 is a path through Cartan geometry [4]:

1. choose a tetrad $q^a{}_\mu$ (the potential prototype);
2. compute the spin connection $\omega^a{}_{\mu b}$;
3. form the torsion $T^a = Dq^a$ and the curvature $R^a{}_b$;
4. read fields by the ECE axioms $A^a = A^{(0)}q^a$ and $F^a = A^{(0)}T^a$.

All of that is local. It works in a laboratory chart. It does not, by itself, say how a spinor closes after a 4π turn, how two field lines can be linked and stay linked, or why a circularly polarised tetrad cannot be written with only two legs.

The Hopf fibration answers those global questions without changing steps 1–4. It is a compact 3-manifold, $S^3 \cong SU(2)$, fibred by circles over an ordinary sphere S^2 [15–17]. The ECE axioms are applied on that manifold exactly as they are applied on a coordinate patch in Volume 1.

This paper does not introduce a second current species (UFT 454–458 already closed that count [6]). It does not re-assert a free-space vacuum $B^{(3)}$ as a Layer-1 field. It does not use device dimensions or unpublished coupling numbers.

Throughout, *Layer 1* means statements that follow from topology, Cartan geometry and the ECE axioms alone; *Layer 2* means proposed physical identifications that still need a calculation or a measurement.

Remark (plain language). Think of Volume 1 as the grammar of the theory: how you go from a frame of four arrows to a force field. This paper is one complete sentence written in that grammar. The sentence is a 3-sphere full of linked circles. Nothing in the grammar is rewritten.

Remark (plain language). Layer 1 is what the mathematics forces; Layer 2 is what we think it means in the laboratory. We keep the two on separate shelves so that a doubt about a laboratory reading never weakens the mathematics.

2 The ECE axioms, written once

Definition 2.1 (Tetrad). A tetrad $q^a{}_\mu$ is a local coframe. Index a is a tangent-space (polarization) index. Index μ is a coordinate index of the base manifold. In UFT 439 the tetrad is often taken diagonal by aligning those two frames.

Definition 2.2 (ECE axioms).

$$A^a = A^{(0)} q^a, \quad (2.1)$$

$$F^a = A^{(0)} T^a, \quad (2.2)$$

$$T^a = Dq^a = dq^a + \omega^a_b \wedge q^b, \quad (2.3)$$

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \quad (2.4)$$

$$DT^a = R^a_b \wedge q^b. \quad (2.5)$$

Equation (2.5) is the first Bianchi identity. It is the homogeneous field equation of the unified theory [5]. The tetrad is a potential, torsion is a field strength. The currents that appear when (2.5) is written in vector form consist of the geometric fields themselves (Volume 1, Eqs. 4.45–4.46 [2]; UFT 456 [10]); they are not external sources.

Remark (plain language). Equations (2.1)–(2.5) say: the frame is the potential, the twist of the frame is the field, and one identity of geometry ties twist to bending. Nothing is put in from outside: even the “charges” and “currents” are fields packed tightly together.

The same skeleton is copied across sectors (Paper 453 [5]):

$$D \wedge T = R \wedge q \quad (\text{geometry}), \quad (2.6)$$

$$D \wedge F = R \wedge A \quad (\text{electrodynamics}), \quad (2.7)$$

$$D \wedge G = R \wedge Q \quad (\text{gravitation / dynamics}), \quad (2.8)$$

$$D \wedge F_{fd} = R \wedge v \quad (\text{fluids / aether}). \quad (2.9)$$

In words. Each line has the same shape. On the left is the change of a field (torsion, the electromagnetic field F , the gravitational field G , the fluid vorticity F_{fd}); on the right the same curvature R is combined with the matching potential (q, A, Q, v). One identity of geometry is read four times.

The ECE wave equation

$$(\square + R) q^a{}_\mu = 0 \quad (2.10)$$

follows from the tetrad postulate¹ (the Evans lemma, Volume 1, Section 2.5.5 [2]; Volume 2, Eq. (11.73) [3]). It appears in several sectors of ECE (gravitation [36], electrodynamics [38], fluid dynamics, and quantum mechanics,

¹The *tetrad postulate* $D_\mu q^a{}_\nu = 0$ states that the complete covariant derivative of the tetrad vanishes; it ties the spin connection ω to the affine connection Γ . Differentiating it once more gives the *Evans lemma*, the origin of Eq. (2.10) [37].

where it is written $(\square + R)\psi = 0$), not only as the wave equation of quantum mechanics [1, 3, 5]. The scalar R in (2.10) is the curvature scalar of Cartan geometry defined through the tetrad postulate; it is not the usual scalar curvature $g^{\mu\nu}R_{\mu\nu}$ of Einstein's theory. No a-priori sign restriction is placed on R . The d'Alembert operator of ECE is $\square = c^{-2}\partial_t^2 - \nabla^2$, corresponding to the Minkowski metric $\text{diag}(1, -1, -1, -1)$ [2, 3]; for static fields $\square \rightarrow -\nabla^2$. Hopf geometry specialises these equations; it does not replace them.

Remark (plain language). The same wave equation shows up wherever ECE describes something that oscillates: light, gravity, flow, and matter waves. The number R in it is a property of the twisted frame, not the curvature number of Einstein's theory, and it may come out positive or negative.

Torsion is not an optional extra. In Riemann geometry the connection is forced to be symmetric and $T = 0$. Cartan geometry drops that restriction [22, 23]. Figure 2.3 is the elementary distinction.

Remark (plain language). Four arrows at a point are the tetrad. If you slide them around a small loop and they come back rotated, that is curvature. If they come back sheared or twisted, that is torsion. ECE says the twist is the field you measure with a coil or a test mass. The four equations above are the same sentence said about light, about falling, about flow, and about the geometry itself.

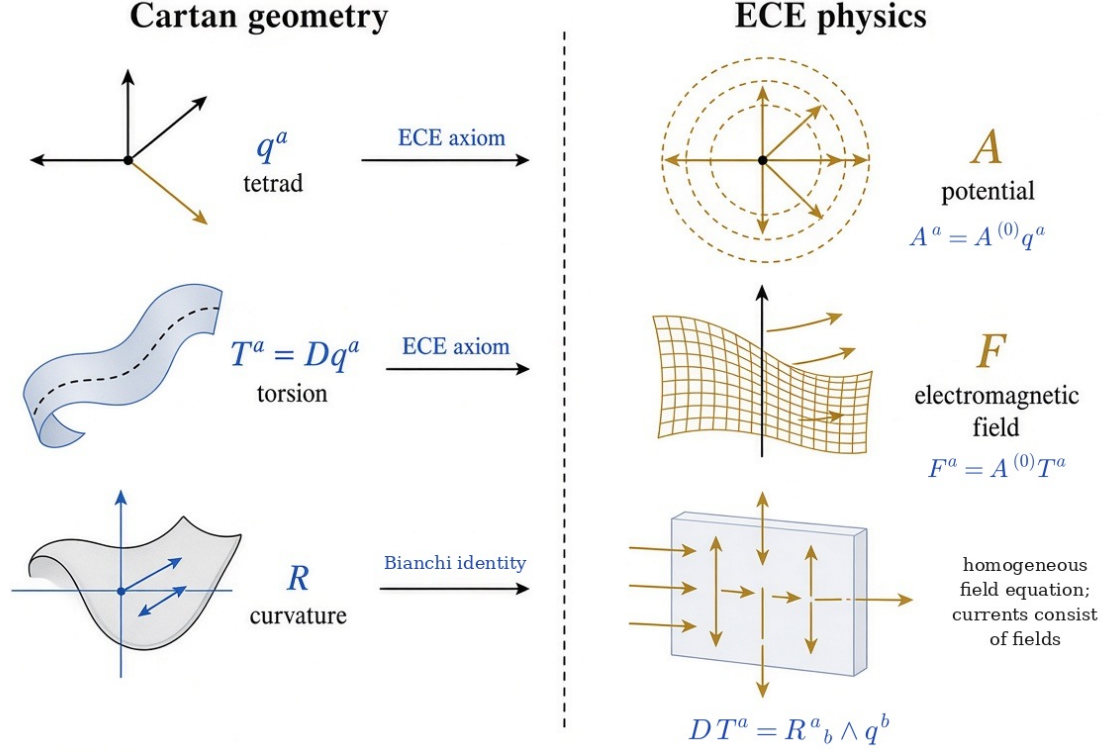


Figure 2.1. The ECE reading of Cartan geometry. Left: tetrad, torsion, curvature. Right: potential $A^a = A^{(0)} q^a$, field $F^a = A^{(0)} T^a$, and the homogeneous field equation given by the first Bianchi identity $DT^a = R^a_b \wedge q^b$. Currents arise from the geometric fields, not as external sources. Each arrow is one axiom or identity, not a new hypothesis.

What the figure shows. The left column shows three objects of Cartan geometry: a frame of arrows (tetrad), a twisted ribbon (torsion) and a bent surface (curvature). The right column shows how ECE reads them: the frame is the potential, the twist is the field, and the bent surface enters only through the first Bianchi identity, which is the homogeneous field equation. Currents are not drawn as an extra input; they are made of the fields.

3 The 3-sphere, quaternions, and $SU(2)$

Definition 3.1 (S^3).

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}. \quad (3.1)$$

This is also the set of unit quaternions and the group $SU(2)$.

In words. S^3 is the set of all pairs of complex numbers whose squared sizes add up to one: a sphere one dimension higher than the surface of a ball. Every point on it can also be read as a rotation.

The covering map

$$SU(2) \longrightarrow SO(3), \quad q \longmapsto (v \mapsto qvq^{-1}) \quad (3.2)$$

is two-to-one. A 2π rotation in ordinary space lifts to a path from q to $-q$ on S^3 . A spinor returns to itself only after 4π . Volume 1 already uses this double cover. The Hopf fibration is that cover written as a fibre bundle, so the extra 2π has a circle you can point at.

ECE field equations

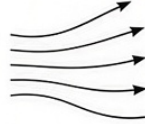
Row 1 Geometry:		$D \wedge T = R \wedge q$
Row 2 Electrodynamics:		$D \wedge F = R \wedge A$
Row 3 Gravitation:		$D \wedge G = R \wedge Q$
Row 4 Fluids:		$D \wedge F_{fd} = R \wedge v$

Figure 2.2. The four sector copies of the first Bianchi identity, after Paper 453. One geometry, four physical readings.

What the figure shows. One and the same identity, “change of a field = curvature $\wedge$ potential”, written four times: for geometry, for electrodynamics, for

gravitation and for fluid flow. This is the unification idea of ECE in a single table.

Remark (plain language). You can turn a belt twice and untwist it; once is not enough. An electron is like that belt. The extra turn lives on a circle that the Hopf map will call a fibre.

4 The Hopf map, written so it can be computed

Definition 4.1 (Hopf map).

$$\pi(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2) \in S^2 \subset \mathbb{R}^3. \quad (4.1)$$

Equivalently $\pi(z_1, z_2) = [z_1 : z_2] \in \mathbb{CP}^1 \cong S^2$.

Remark (plain language). The Hopf map takes four numbers on the 3-sphere and returns a point on an ordinary globe. All the points that differ only by an overall phase land on the same spot; that set of points is one circle, the fibre.

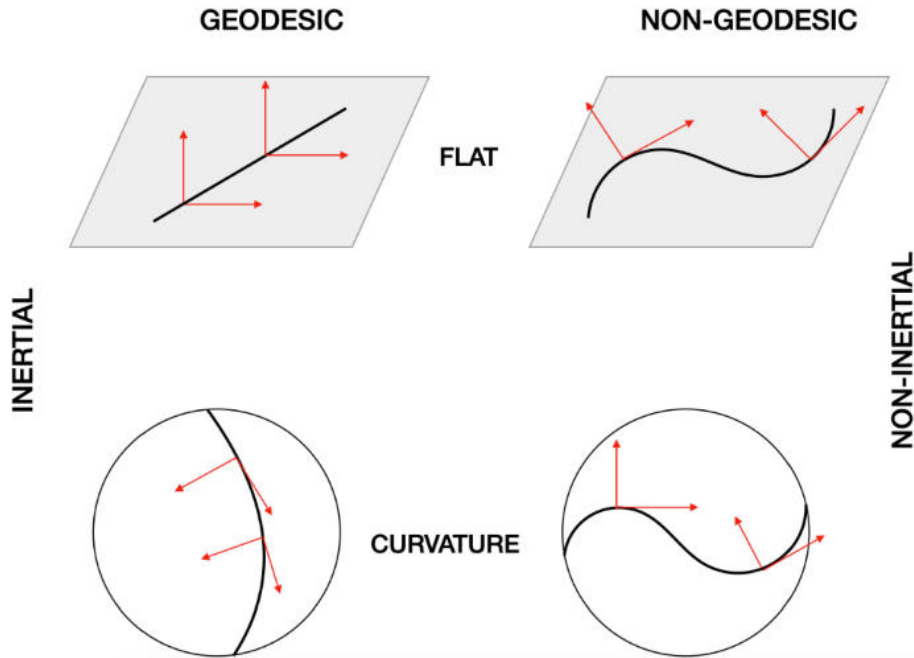


Figure 2.3. Curvature turns a geodesic. Torsion twists the frame as it is carried (non-geodesic, non-inertial panel). ECE uses both. The labels are the geometric ones of Volume 1, not a claim about a particular metric theory.

What the figure shows. Top row: flat paper. A straight path keeps its frame (left); a curved path drags a frame that twists (right). Bottom row: a sphere.

Even a “straight” great-circle path turns its frame because the surface is curved (left); a non-geodesic path on the sphere shows both effects (right). Curvature and torsion are different things, and ECE uses both.

Proposition 4.1 (Fibres). $\pi^{-1}(n)$ is a great circle. Distinct fibres do not meet. The bundle is a principal $U(1)$ bundle of Chern class $c_1 = 1^2$.

Stereographic projection³ from one pole of S^3 onto $\mathbb{R}^3$ turns those great circles into ordinary circles (or one straight line). That is all the colourful figures are.

Remark (plain language). Take a globe. Over each city sits one circle in a higher sphere. Every other circle goes through that one exactly once. The drawings are shadows of that fact, obtained by a standard projection from four numbers down to three.

Remark (plain language). The Chern class $c_1 = 1$ counts how much the circles are twisted around each other as you go once around the globe: exactly one full twist. It is a whole number, so it cannot be changed by smooth deformation.

²A *principal $U(1)$ bundle* is a space that looks locally like $(\text{base}) \times (\text{circle})$, with the circle acting as a phase. The *Chern class* c_1 is an integer that counts how often the circles twist as one goes once around the base; for the Hopf bundle $c_1 = 1$ [15, 16, 25].

³Stereographic projection maps a sphere minus one point onto flat space by drawing straight lines from that point. It maps circles to circles (or to straight lines through infinity) [17].

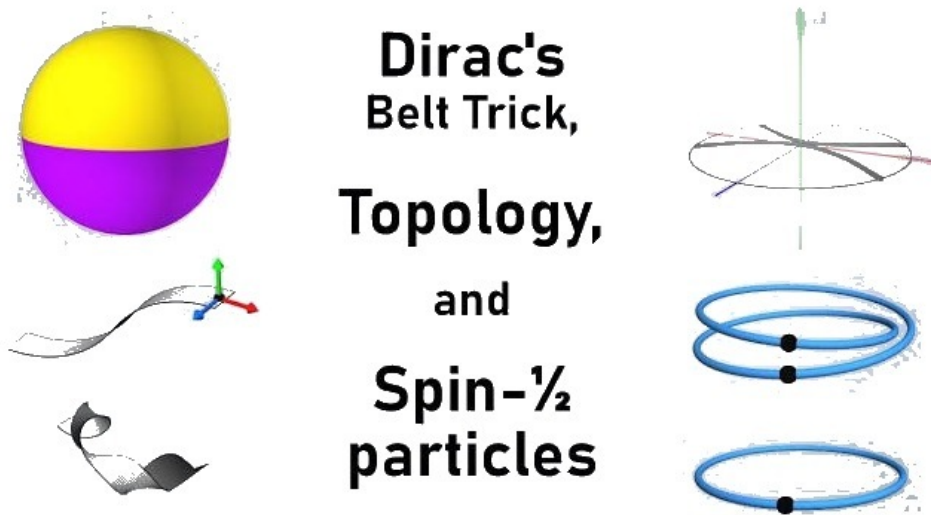


Figure 3.1. The 4π story in pictures: Dirac belt, linked rings, and the double cover $SU(2) \rightarrow SO(3)$. This is the same S^3 used below. After standard topology lectures.

What the figure shows. Dirac's belt trick. A belt twisted by one full turn (2π) cannot be straightened without moving its ends, but a belt twisted by two full turns (4π) can. The rings on the right show the same fact as a double loop. Spin- $\frac{1}{2}$ particles behave like the belt.

5 Linking number: the integer that ECE helicity already wants

Before the two integrals of this section are written, their three symbols are defined.

Definition 5.0 (Contact form, vector potential and magnetic field).

- α is the standard contact 1-form on S^3 : the connection form of the Hopf bundle along the S^1 fibre. In the Euler/Hopf angles of Section 13, $\alpha = e^3 = \frac{1}{2}(d\psi + \cos\theta d\phi)$; it satisfies $\alpha(R_\alpha) = 1$ on the fibre generator $R_\alpha = 2\partial_\psi$, and $d\alpha$ is proportional to the pulled-back area form of the base. In Definition 5.2 α is normalised (and oriented) so that $d\alpha = \pi^*\omega_{S^2}$ with unit total area; for $\alpha = e^3$ itself the raw integral is $\int \alpha \wedge d\alpha = -4\pi^2$ (Section 17 item 1). The full geometric definition and the contact condition $\alpha \wedge d\alpha \neq 0$ are given in Section 6 (computed in Section 13.3).
- $\mathbf{A}$ is the vector potential in $\mathbb{R}^3$ obtained by pushing α forward under stereographic projection of S^3 from one pole (Section 13.8).

- $\mathbf{B} = \nabla \times \mathbf{A}$ is the corresponding magnetic field (field strength) in $\mathbb{R}^3$; it is the vector form of $d\alpha$.

Remark (plain language). α is a rule that says “how far along the circle” you move. Flattened into ordinary space it becomes a vector potential $\mathbf{A}$, and its curl is the field $\mathbf{B}$. The two integrals below count the same twist, once on the sphere and once in the laboratory picture.

Definition 5.1 (Hopf link). Any two distinct fibres form a Hopf link: two unknots with linking number 1.

Definition 5.2 (Hopf invariant). If $d\alpha = \pi^*\omega_{S^2}$ and $\int_{S^2} \omega_{S^2} = 1$, then

$$H(\pi) = \int_{S^3} \alpha \wedge d\alpha = 1. \quad (5.1)$$

Magnetic helicity⁴

$$\mathcal{H} = \int \mathbf{A} \cdot \mathbf{B} \, d^3x \quad (5.2)$$

is the same integral in vector clothing. ECE already conserves helicity when the Beltrami condition $T = \kappa q$ holds. The Hopf invariant supplies the integer that κ multiplies.

Remark (plain language). Helicity measures how much field lines wind around each other. On the Hopf bundle that winding is not a measured decimal but a whole number, one.

⁴Magnetic (or fluid) helicity measures the linking of field lines; its topological meaning was established by Moffatt [19] and placed in the Hopf framework by Whitehead’s integral formula for the Hopf invariant [18]; see also Arnold and Khesin [20].

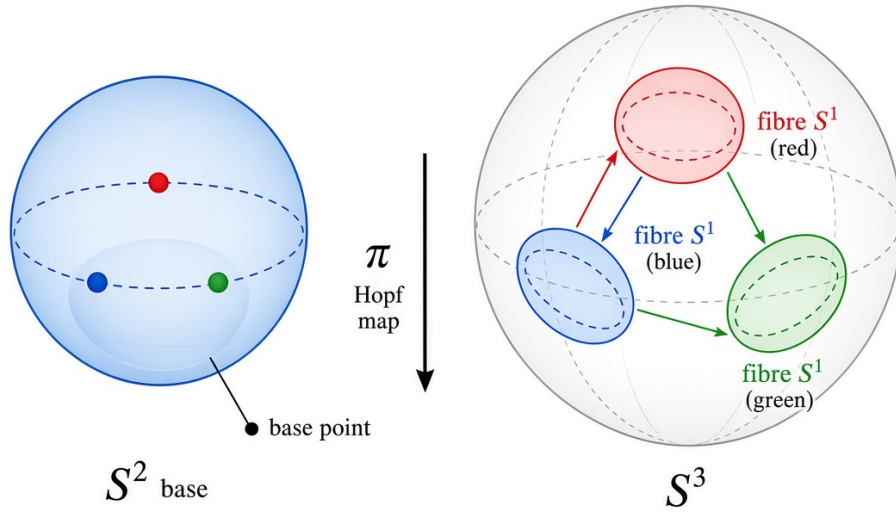


Figure 4.1. The map in one picture: base points on S^2 and fibres on S^3 . Generated as a teaching schematic. The physical map is $S^3 \rightarrow S^2$; fibres sit over points.

What the figure shows. Each coloured point on the ordinary sphere S^2 (left) is the shadow of a whole circle of the same colour in the 3-sphere S^3 (right). Different points give different circles, and the circles never touch.

Remark (plain language). Can you pull the two rings apart without cutting? No. How many times do they go through each other? Once. That number is what $\int \mathbf{A} \cdot \mathbf{B}$ is counting when the field lines are these rings.

6 Contact form, spin connection, and why $T^a \neq 0$

Choose the standard contact form α on S^3 (the connection form along the S^1 fibre, Definition 5.0). Then

$$\alpha \wedge d\alpha \neq 0. \quad (6.1)$$

The Reeb field of α is tangent to the fibres⁵.

⁵For a contact form α the *Reeb field* R_α is the unique vector field with $\alpha(R_\alpha) = 1$ and $\iota_{R_\alpha} d\alpha = 0$. Here ι_X (Greek iota) is the *interior product*: it inserts the vector X into the first slot of a form, $(\iota_X d\alpha)(Y) = d\alpha(X, Y)$. The condition therefore says $d\alpha(R_\alpha, Y) = 0$ for every vector Y . Contact forms and their Reeb fields are the natural language for Beltrami fields [21].

Remark (plain language). A contact form is a field of tiny planes that twist as you move, so that you can never lay them flat into a stack of sheets. Equation (6.1) is the test: if the planes could be stacked, the product would be zero. The Reeb field is the direction that sticks out of the planes; here it runs along the fibres.

On this coframe the ECE spin connection ω^a_b has a piece along the fibre equal to α . The first structure equation then produces a torsion 3-form whose contact piece is $\alpha \wedge d\alpha$. Because $c_1 = 1$, that piece cannot be gauged to zero on the whole total space.

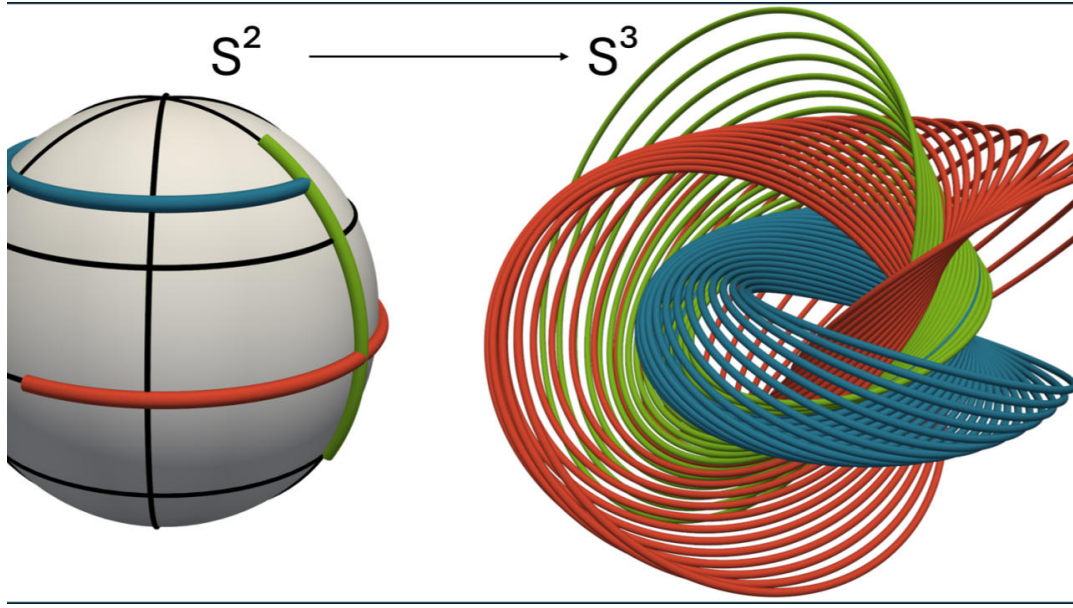


Figure 4.2. The same correspondence after stereographic projection into $\mathbb{R}^3$. Colour of a circle equals colour of its base point. The printed arrow on the source graphic runs the wrong way.

What the figure shows. The circles of Figure 4.1 drawn in ordinary space after stereographic projection. They remain circles, every pair is linked, and the colour of each circle still matches its base point.

This is the global reason for the local remark of UFT 439 and Volume 1: spin connections always exist, because torsion is present by geometry. On the Hopf bundle the remark is a theorem about Chern class, not a modelling preference.

The antisymmetry constraints of Volume 1, Section 5.1, still close the system. No additional current species is introduced; the Hopf coframe does not create a second space. One physical space [6, 2].

Remark (plain language). If you try to put a consistent phase arrow on every linked circle at once, a twist is left over. ECE already named that

twist “torsion” and “spin connection”. The Hopf bundle is a place where you cannot set the twist to zero and still keep the spinor. ECE keeps its two currents, both made of fields; the Hopf picture adds none.

7 Alignment with UFT 439: the third tetrad leg and $B^{(3)}$

UFT 439 computes the full Cartan path for a circularly polarised electromagnetic wave. The tetrad must remain non-singular. Therefore at least one third component A^a_3 is present. UFT 439 notes that a constant magnetic field in the Z direction appears, “which in principle is the $B^{(3)}$ field of Evans” [4]⁶.

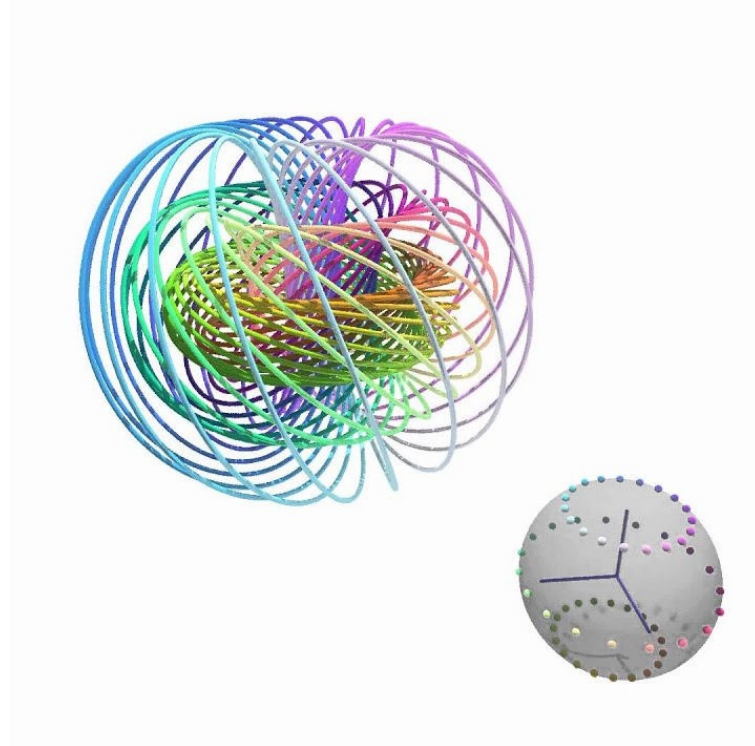


Figure 4.3. A denser filling. Each ring is one fibre. The small sphere is the base S^2 . After Niles Johnson [34].

What the figure shows. Many fibres drawn at once. Nested, mutually linked rings fill the whole of space; the small globe at lower right marks the base points that the rings sit over.

⁶The $B^{(3)}$ field is the longitudinal magnetic field of circularly polarised radiation proposed by Evans in 1992 [31]. Its status as a free-space field is disputed; this paper does not use it as a Layer-1 object.

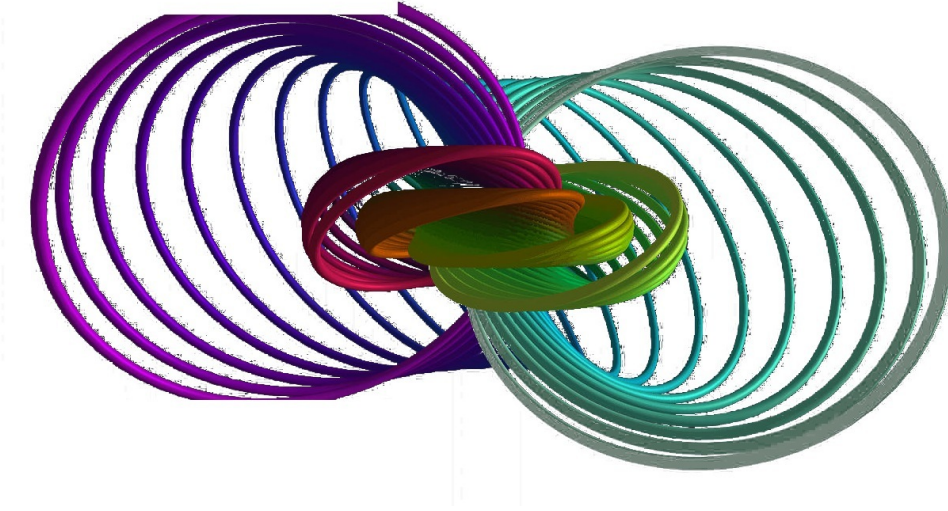


Figure 4.4. Nested Clifford tori. A one-parameter family of fibres lies on $S^1 \times S^1 \subset S^3$. After PhiloGB [35].

What the figure shows. Fibres sitting over one circle of latitude on the globe sweep out a doughnut-shaped surface (a Clifford torus). Different latitudes give tori nested inside one another; each ring winds once around each torus.

The Hopf picture of that sentence is this. A circularly polarised wave is a rotating frame in the xy -plane. The rotation generator is an $SO(2) \subset SO(3)$ action. The double cover lifts it to a motion along an S^1 fibre of S^3 . The third tetrad leg is the leg dual to that fibre. You do not add $B^{(3)}$ by hand. You refuse to write a singular tetrad.

Layer-1 statement: a circularly polarised tetrad on the Hopf bundle has a non-vanishing third leg. That is UFT 439 plus $c_1 = 1$.

Layer-2 statement, not promoted: the Inverse Faraday Effect⁷ is the matter-mediated reading of that third leg. In empty space with residual connection $\delta\omega = 0$, Maxwell theory is recovered and the Rañada hopfion lives in that sector.

⁷The Inverse Faraday Effect is the static magnetisation induced in a material by circularly polarised light, first observed by van der Ziel, Pershan and Malmstrom [32]. It requires matter.

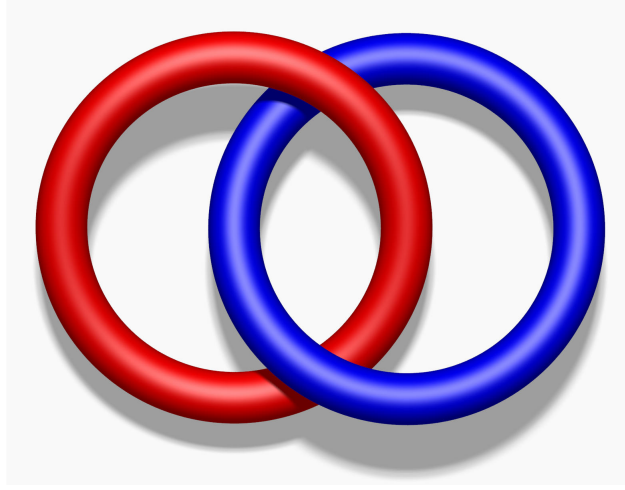


Figure 5.1. The Hopf link. This is the elementary observable of the construction: an integer you can count.

What the figure shows. The simplest picture of the whole construction: two rings that pass through each other exactly once. That count, the linking number 1, is the integer of Eq. (5.1).

This split is written so that Bruhn-type objections⁸ to a free-space $B^{(3)}$ do not touch the tetrad calculation of UFT 439.

Remark (plain language). The residual $\delta\omega$ is the part of the spin connection left over after the ordinary (Levi-Civita) part is taken out. Where it is zero, as in empty space, the equations reduce to ordinary Maxwell theory. Where matter is present it need not be zero, and that is where an effect like the Inverse Faraday Effect can live.

Remark (plain language). A spinning light wave is a frame that turns. A frame with only two arrows becomes flat at some instant and the maths breaks. UFT 439 keeps a third arrow so the frame never dies. That third arrow is what people argued about under the name $B^{(3)}$. Here we keep the UFT 439 calculation and postpone the laboratory name.

8 Alignment with UFT 454–458: potential flux V

Nomenclature follows UFT 454–458.

⁸Published critiques of ECE and of a free-space $B^{(3)}$, e.g. Bruhn, Hehl and Jadczyk [33]. Keeping the Inverse Faraday Effect in Layer 2 means those critiques do not bear on the Layer-1 tetrad calculation.

Definition 8.1 (Potential flux, after UFT 454–458). V is the potential-flux area density (the “area flux density of a potential”, units V m^{-2} [10]), the counterpart of the electronic current density J . Together with the potential density ρ_p it forms the homogeneous charge–current pair of ECE, dual to (ρ_e, J) . It replaces the older reading of the engineering-model slides, in which a “homogeneous current” was attributed to curvature; like J , V consists of fields. I_H is the true potential current constructed from V . V has the unit V m^{-2} (in flow units $\text{V s m}^{-2} \text{s}^{-1}$); volts and amperes are dual, as J has the unit A m^{-2} . The counterpart of ρ_e is ρ_p , with units V s m^{-3} .

In the symmetrised vector form of UFT 456, Eqs. (20)–(23) [10], the ECE field equations read

$$\nabla \cdot \mathbf{B} = \rho_p, \tag{8.1}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{V}, \tag{8.2}$$

$$\nabla \cdot \mathbf{D} = \rho_e, \tag{8.3}$$

$$-\partial_t \mathbf{D} + \nabla \times \mathbf{H} = \mathbf{J}. \tag{8.4}$$

In UFT 456, Eqs. (28)–(31), all four source terms ρ_p , $\mathbf{V}$, ρ_e , $\mathbf{J}$ are then expressed by the fields $\mathbf{E}$ and $\mathbf{B}$ multiplied by wave-number vectors κ , $\kappa^{(\Lambda)}$ that are determined by the curvature and torsion of spacetime: “In ECE theory, all charge and current terms are expressed by fields” [10]. This is the vector form of Volume 1, Eqs. (4.45)–(4.46) [2].

Remark (plain language). Equations (8.1)–(8.4) look like Maxwell’s, with one change: the magnetic pair has its own charge ρ_p and current $\mathbf{V}$, mirror images of the electric pair. And none of the four sources is put in by hand; each is a field multiplied by a twist rate of space.

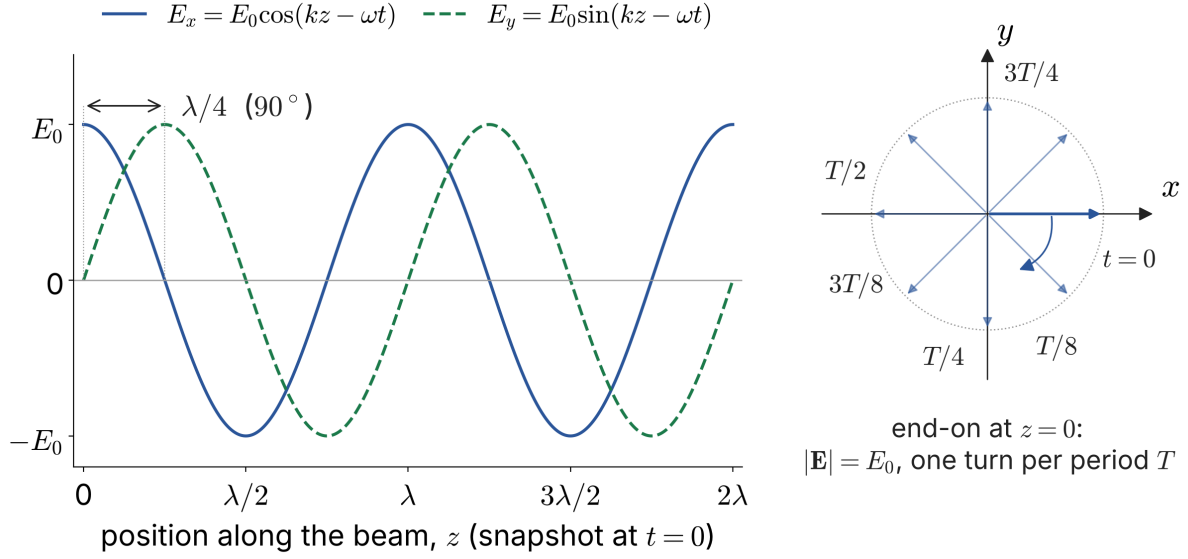


Figure 7.1. Circular polarisation of a travelling wave. Left: snapshot at $t = 0$ of the two transverse components $E_x = E_0 \cos(kz - \omega t)$ and $E_y = E_0 \sin(kz - \omega t)$; their crests are shifted by a quarter wavelength, a 90° phase difference. Right: the field seen end-on at $z = 0$; the electric vector keeps its length and turns once per period T . UFT 439 asks what tetrad can carry this rotation without becoming singular. Drawn for this paper.

What the figure shows. Two equal waves, one pointing along x and one along y , are shifted by a quarter of a wavelength. Added together they give an electric field of constant length that turns steadily: seen end-on it sweeps round a circle once per period, and along the beam its tip traces a corkscrew (shown in Figure 7.2).

Facts used here from UFT 454–458, without change:

- Spin connections always exist; torsion is present by geometry.
- There is one physical space. V is a density wave (pressure construct) of space itself, not a massive carrier.
- J expands toward the wire surface. V concentrates. Energy can flow from the environment into V , which is the thermodynamic origin of a local temperature drop. Ohmic J heats.
- After two B fields cancel, the surviving object of the extended Faraday law is primarily V . Sharp dB/dt flanks make V visible. V can appear without a bifilar cancel.
- Antisymmetry constraints are those of Cartan geometry and the tetrad postulate (Volume 1, Section 5.1). No additional current species is introduced; J and V , both made of fields, are the whole count.

Hopf alignment, proposed as a calculation to be checked in Maxima, not as a finished theorem:

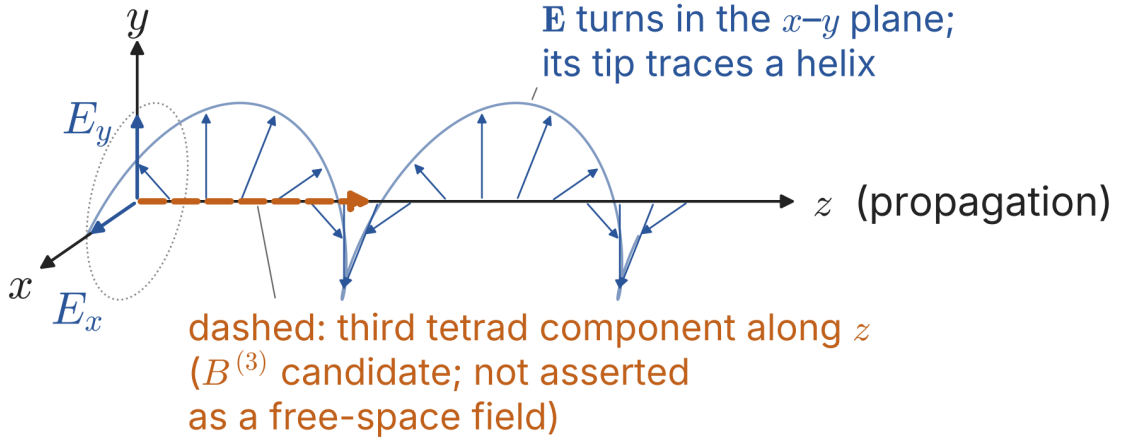


Figure 7.2. The same rotation in three dimensions. The transverse axes x and y span the plane in which E_x and E_y rotate; z is the direction of propagation. The dashed arrow along z is the candidate third component. In this paper it is not asserted as a free-space vacuum field. It is the third tetrad leg required by non-singularity. Whether a laboratory magnetization appears is a matter-mediated residual question (Section 14).

What the figure shows. The electric field vector turns in the x - y plane, perpendicular to the direction of travel z , and its tip traces a helix. The dashed arrow points along z , at right angles to both transverse axes: it is the third tetrad component needed to keep the frame non-singular, not a claimed free-space magnetic field.

The connection form α along the Hopf fibre is a $U(1)$ potential on S^3 . Its exterior derivative $d\alpha$ is the area form pulled back from S^2 . In a laboratory chart the 3-vector part of α is a candidate geometric prototype for the potential that generates V . Because $\alpha \wedge d\alpha \neq 0$, the flux is linked. That would give V an integer helicity, which is a testable distinction from ohmic J .

This is an alignment, not a device recipe. No winding numbers of laboratory coils are given.

Remark (plain language). UFT 454–458 say: besides the current of electrons there is a current of potential itself, a pressure-wave of space. Electrons spread to the skin of a wire and heat it. The potential-wave tightens and can cool the neighbourhood. The Hopf circle is offered as the shape that wave might prefer, because that circle is already the phase of the spinor. Whether laboratory V really follows those circles is a measurement, not a slogan.

9 Alignment with m-theory (UFT 449–451)

m-theory writes a radial metric function $m(r)$ and then runs the UFT 439 path: tetrad, spin connection, torsion, fields [7]. Two results matter here.

First: a central force appears that is generated by the geometry itself, with no extra source term.

Second: a rotational field appears even when the potential has no rotational part. In the rotating-spacetime paper (UFT 451) six basic geometric forms of force and curl are listed, and UFT 451 concludes that charge and spin of elementary particles can originate in the geometry.

Hopf alignment: a rotational field with no rotational potential is exactly what a fibre-wise connection produces. The potential can be a function of the base S^2 only; the connection along S^1 still generates a linked B -like torsion. Particle spin as geometry is the 4π circuit of Section 3. Particle charge as geometry is the Chern class $c_1 = 1$ (one unit of fibre twist). Both sentences are Layer-1 topology plus the ECE axioms. Mass numbers are not derived in this paper.

Remark (plain language). m-theory already found a swirl in the maths when no swirl was put in by hand. The Hopf fibre is a swirl that lives in the extra circle, so you do not have to put it into the potential on ordinary space.

10 Beltrami fields and the hopfion

On round S^3 the lowest constant-norm curl eigenfield is the Hopf field⁹, $\text{curl } V = 2V$. UCRC-CG [40] writes the same condition as $T = \kappa q$. The Rañada hopfion is the pull-back of the S^2 area form by a time-dependent Hopf map; it is an exact null solution of vacuum Maxwell theory whose field lines are Hopf links [8, 9, 26]; linked and knotted light beams of this kind have since been constructed explicitly [27].

In the residual rule of this paper the hopfion lives in the $\delta\omega = 0$ sector: ordinary Maxwell, integer helicity, no claimed free-space $B^{(3)}$.

Remark (plain language). A Beltrami field is one whose curl points the same way as the field itself, like a corkscrew whose twist follows its own axis. The condition $T = \kappa q$ says the same thing about torsion and the tetrad: the twist of the frame lies along the frame.

⁹A *Beltrami field* satisfies $\nabla \times \mathbf{u} = \lambda \mathbf{u}$: its curl is parallel to itself. On the round 3-sphere the Hopf field is the simplest example [20, 21].

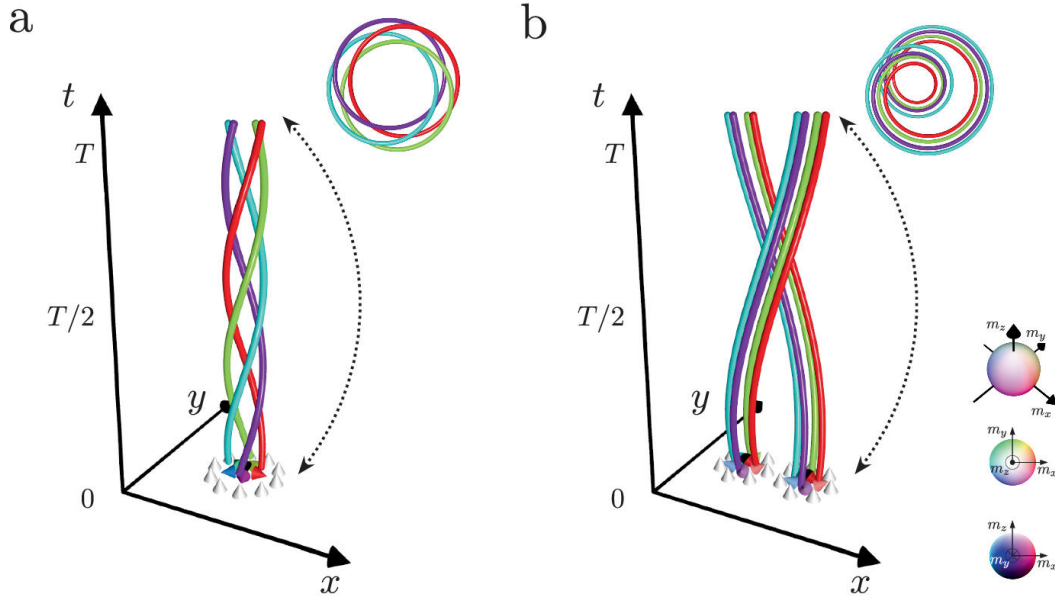


Figure 10.1. Linked world-lines of a hopfionic texture in spacetime. Topology only. Not a coil drawing.

What the figure shows. Field lines of a hopfion followed through time (vertical axis). The bundles twist around each other but stay linked: the topology is preserved while the configuration evolves.

Remark (plain language). There is an exact radio-wave solution whose field lines are the linked rings of Figure 5.1. It has been in the literature since 1989. ECE reads it as a compact piece of torsion with helicity one.

11 Spinors and the Bloch sphere

A pure qubit is a point of S^3 . Physical rays are $S^3/S^1 = S^2$. That quotient is the Hopf map. Spinors are handled in Volume 2 of the textbook [3], where the ECE Fermion equation is derived (Chapter 11), both from the Dirac equation and by factorising the ECE wave equation (2.10); it is equivalent to the Dirac equation. The classical bundle already contains the double cover, so the quantum volume does not need a new topology.

Remark (plain language). The arrow on a Bloch sphere is a city on the globe. The spinning object itself is a point of the higher sphere. The extra circle is phase. Same extra circle as in Section 4. The equation that moves such spinors is the ECE Fermion equation of Volume 2; this paper only supplies the space they live on.

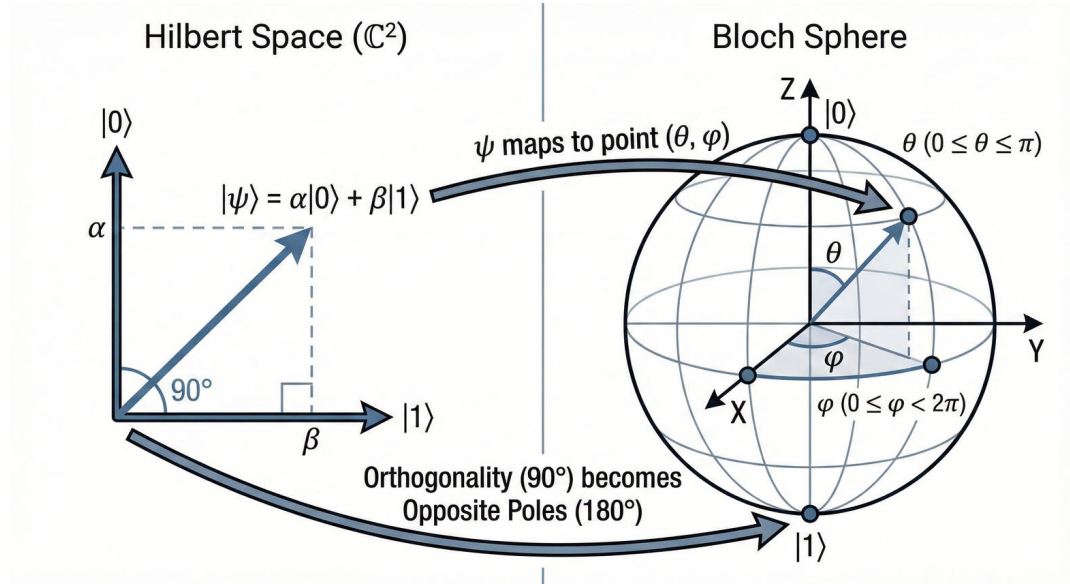


Figure 11.1. Hilbert space $\mathbb{C}^2$ to Bloch sphere S^2 . Orthogonal states become antipodes. After Punnett.

What the figure shows. A two-state quantum system is a pair of complex amplitudes (left). Dropping the overall phase maps it to a single point on the Bloch sphere (right). States at right angles in the left picture become opposite poles on the sphere. This map is the Hopf map.

12 Clifford algebra as bookkeeping, not a new axiom

The UPITEC note of 19 January 2024 and UFT 456 state the rule used here [10, 11]. A. Goel wrote electrodynamics in Clifford algebra. UFT 456 then rewrote the pieces that matter for ECE in ordinary vector form, so a reader of Volume 1 does not have to learn a second language first. Clifford algebra is therefore a transcription, complementary to Cartan forms. It does not replace the tetrad path of UFT 439.

Definition 12.1 (Geometric product). For vectors a, b in a real quadratic space,

$$ab = a \cdot b + a \wedge b. \quad (12.1)$$

The first summand is a scalar (grade 0). The second is a bivector (grade 2): an oriented area. The algebra generated by this product is the Clifford algebra of that space.

Remark (plain language). Multiplying two arrows in Clifford algebra gives two things at once: how much they point the same way (a number) and

the area they span (a tile). Keeping both is what makes the duality between J and V short to write.

Definition 12.2 (Grades in three dimensions). $Cl(3, 0)$ has eight basis pieces: 1 (scalar), three vectors e_i , three bivectors $e_i e_j$, and one pseudoscalar $I = e_1 e_2 e_3$ with $I^2 = -1$. The unit imaginary of laboratory electromagnetism can be read as this I , not as an extra scalar $\sqrt{-1}$.

Remark (plain language). A grade is the “dimension” of a piece: 0 for a number, 1 for an arrow, 2 for a flat tile, 3 for a little box. The box squared gives -1 , which is why it can play the role of the imaginary unit.

Basis element	Grade	Multiplicity
1	0 (scalar)	$\binom{n}{0} = 1$
$e_1, e_2, e_3, \dots, e_n$	1 (vector)	$\binom{n}{1} = n$
$e_1 e_2, e_1 e_3, \dots, e_1 e_n, \dots, e_2 e_3, \dots, e_2 e_n, \dots, e_{n-1} e_n$	2 (bivector)	$\binom{n}{2} = \frac{n(n-1)}{2}$
$e_1 e_2 e_3, e_1 e_2 e_4, \dots, e_1 e_2 e_n, \dots, e_{n-2} e_{n-1} e_n$	3 (trivector)	$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$
...	...	...
$e_1 e_2 e_3 \dots e_n$	n (pseudoscalar)	$\binom{n}{n} = 1$

Figure 12.1. Basis elements of a Clifford algebra by grade. After a standard geometric-algebra teaching table.

What the figure shows. A Clifford algebra built on n directions has one scalar, n vectors, $n(n-1)/2$ bivectors and so on, one pseudoscalar at the top. The counts are the binomial coefficients.

Definition 12.3 (The nesting used in this paper).

$$Cl(0, 1) \subset Cl(0, 2) \subset Cl(0, 3) \subset Cl(1, 3). \quad (12.2)$$

In order: complex phases; quaternions $\mathbb{H}$; the algebra generated by the three spatial unit vectors of STA, which square to -1 in the Minkowski signature $(+, -, -, -)$ of ECE; spacetime algebra (STA) [28, 29]. Each algebra in the chain is generated by a subset of the generators of the next. The Pauli algebra $Cl(3, 0)$ of Definition 12.2, used for laboratory 3-vectors, sits inside $Cl(1, 3)$

as its even subalgebra (the spacetime split), not as a member of this chain. The Hopf total space is the first interesting step:

$$S^3 \cong \{q \in \mathbb{H} : |q| = 1\} \cong \text{Spin}(3) \subset Cl(0, 2). \quad (12.3)$$

Fibre multiplication $z \mapsto e^{I\theta}z$ is the Clifford product by a unit complex (or quaternionic) phase.

In words. Each algebra in the chain adds one more direction whose square is -1 : one gives the complex numbers, two give the quaternions, three give the spatial part of spacetime algebra, and adding time gives the full spacetime algebra.

Remark (plain language). A Clifford number is a pile of four kinds of object: a size, some arrows, some flat tiles, and a box. You multiply the pile as one thing. The extra circle of the Hopf picture is the smallest such pile that can turn a spinor: the unit quaternions.

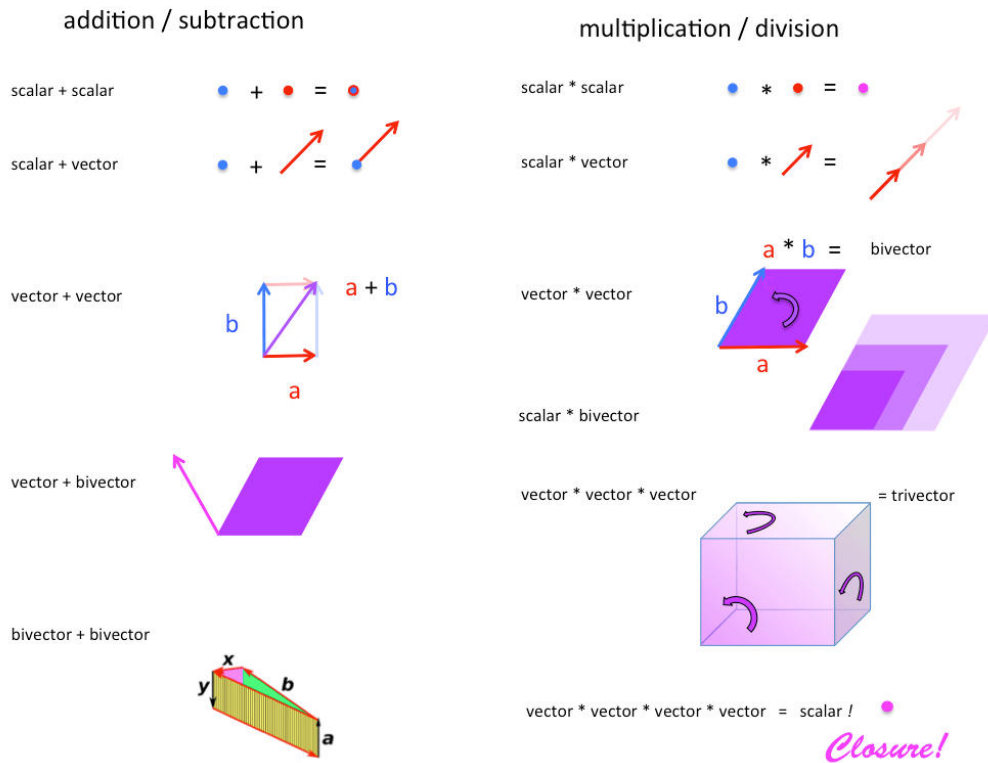


Figure 12.2. Addition and the geometric product, drawn by grade. A product of two vectors is a number plus an area. After Slehar's visual introduction.

What the figure shows. Left: adding pieces of different grade simply collects them side by side. Right: multiplying vectors produces numbers, oriented areas and oriented volumes; four vectors multiplied in three dimensions close back to a number.

12.1 Spacetime algebra and one Maxwell equation

In STA the Faraday field is a single bivector

$$F = \mathbf{E} + Ic\mathbf{B}, \quad (12.4)$$

and the four Maxwell equations collapse to

$$\nabla F = J, \quad (12.5)$$

with J a spacetime current [12, 13]. Vacuum is $\nabla F = 0$. Electric–magnetic duality is multiplication by the pseudoscalar I . Helicity is the leftover global phase of F . A circularly polarised wave is an eigenbivector of I .

This is a rewrite of vacuum Maxwell theory. It is Layer 1 mathematics. It is not yet ECE, because ECE still reads $F^a = A^{(0)}T^a$.

Remark (plain language). In this language electric and magnetic fields are two parts of one object, and the four Maxwell equations are one equation. It is the same physics written more compactly; it does not yet contain the ECE twist.

12.2 Why the Clifford route runs through Goel’s work

The UPITEC note of January 2024 and UFT 456 record four usable facts [11, 10].

1. Maxwell first wrote the field equations in quaternions. Heaviside cut them to three-vectors and lost some dual pairings.
2. In Clifford algebra two sorts of E and of B appear: a perpendicular coupling and a parallel coupling. The parallel sector is the Beltrami sector $T = \kappa q$ already used in this paper.
3. Symmetric electric and magnetic current densities, which ECE already writes, produce extra terms in the field energy. UFT 456 identifies one of those terms with the Heaviside flow outside a conductor. That identification is a physical reading of a grade, not a coil recipe.
4. Dualities of J and V are shorter in Clifford form. ECE in vector form is not wrong and gives the same results [10].

Proposition 12.1 (Proposed transcription of V). Write one multivector source $\mathcal{J}$. Its grade-1 piece is the carrier current J . The Hodge / pseudoscalar

dual grade is the candidate for the potential-flux area density V of UFT 454–458. Units must still be checked against UFT 457. This is Layer 2 until that check is done against UFT 457.

The method of UFT 456 is binding: write the vector form first, then note the grade. A reader of Volume 1 is never required to compute with 16-component multivectors.

12.3 What this closes on the UFT list

- J versus V : two faces of $\mathcal{J}$, one space.
- Circular tetrad of UFT 439: F an eigenbivector of I ; the third tetrad leg is the remaining spatial direction that keeps the rotor invertible.
- Contact form α on S^3 : a left-invariant quaternionic 1-form. $\alpha \wedge d\alpha$ is a grade-3 piece of the same algebra.
- Wave equation: in STA the operators $\nabla\psi = 0$ and $\nabla F = 0$ are one first-order operator acting on different grades; the spinor case is the ECE Fermion equation of Volume 2 [3].
- Comparison note, not an identification: Gauge Theory Gravity (Lasenby–Doran–Gull; Hestenes) also uses a tetrad-like gauge and a rotation gauge with torsion sourced by spin [14]. It is the closest mainstream cousin of the ECE Cartan path. It does not set $F \propto T$.

What Clifford does not do in this paper: derive a mass number, introduce a second space, or supply device dimensions. Evans’ internal note on a Pauli-spinor identification (“Clifford paradox”) is left aside; Layer 1 does not rest on it.

Remark (plain language). UFT 456 asks for a language in which “current of electrons” and “current of potential” look like two sides of one object. Clifford algebra is that language. We still compute the UFT 439 way: tetrad, spin connection, torsion, field. We only name the grade when the duality would otherwise take a page.

13 Computed alignment identities

The identities below were derived in SymPy from the Hopf coframe written in Euler / Hopf angles $(\theta, \phi, \psi) \in [0, \pi] \times [0, 2\pi] \times [0, 4\pi]$. The script is `hopf_alignment_sympy.py` (this folder). No laboratory coupling is used. Every item is Layer 1 unless tagged otherwise.

13.1 Orthonormal coframe

The round metric on S^3 is

$$ds^2 = \frac{1}{4} (d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2). \quad (13.1)$$

The orthonormal coframe is

$$e^1 = \frac{1}{2} d\theta, \quad (13.2)$$

$$e^2 = \frac{1}{2} \sin \theta d\phi, \quad (13.3)$$

$$e^3 = \frac{1}{2} (d\psi + \cos \theta d\phi). \quad (13.4)$$

The third leg e^3 is the contact form α . It depends on the base coordinate θ and on the fibre coordinate ψ . Its ϕ -component is the Dirac-monopole potential on S^2 .

Remark (plain language). These three formulas are the tetrad of the 3-sphere: two arrows on the globe and one along the circle. The third one is the contact form of Section 5.

13.2 Fibre invariance of the Hopf map

With $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ and

$$\pi(z) = (2(x_1x_3 + x_2x_4), 2(x_2x_3 - x_1x_4), |z_1|^2 - |z_2|^2), \quad (13.5)$$

SymPy simplifies

$$\pi(e^{i\vartheta} z) - \pi(z) = 0 \quad (13.6)$$

identically. The S^1 action is a fibre motion.

In words. Multiplying both complex coordinates by the same phase $e^{i\vartheta}$ moves a point around its fibre circle but leaves its shadow on the globe exactly where it was.

13.3 Contact condition

Exterior calculus on the coframe gives the identities

$$d\alpha = -2 e^1 \wedge e^2, \quad (13.7)$$

$$\alpha \wedge d\alpha = -2 e^1 \wedge e^2 \wedge e^3 \neq 0. \quad (13.8)$$

Explicitly, $d\alpha = -\frac{1}{2} \sin \theta d\theta \wedge d\phi$ and $\alpha \wedge d\alpha = -\frac{1}{4} \sin \theta d\theta \wedge d\phi \wedge d\psi$. The contact condition of Section 6 is therefore a computed identity, not a citation.

In words. The derivative of the fibre form is an area on the globe. Multiplying it by the fibre form gives a non-zero volume, so the planes of the contact structure really do twist.

13.4 Reeb field

The vector field $R_\alpha = 2\partial_\psi$ satisfies $\alpha(R_\alpha) = 1$ and $\iota_{R_\alpha} d\alpha = 0$, i.e. $d\alpha(R_\alpha, Y) = 0$ for every vector Y (ι is the interior product). So R_α is the Reeb field of α . It is the third tetrad vector dual to e^3 . This is the UFT 439 third leg, written on S^3 . (The Reeb field is written R_α here so that it is not confused with the curvature scalar R .)

In words. R_α is the arrow that points along each fibre with unit length as measured by α ; flowing along it leaves the twisting of the planes unchanged.

13.5 Teleparallel torsion

If the spin connection is set to zero on this coframe (Weitzenböck / teleparallel choice¹⁰), the first structure equation collapses to $T^a = de^a$:

$$T^1 = 0, \tag{13.9}$$

$$T^2 = \frac{1}{2} \cos \theta d\theta \wedge d\phi, \tag{13.10}$$

$$T^3 = d\alpha = -\frac{1}{2} \sin \theta d\theta \wedge d\phi. \tag{13.11}$$

Torsion is not identically zero. T^3 is the area form of the base S^2 .

Remark (plain language). Teleparallel means: declare the frame itself to be “not rotating”. Then every bit of twist shows up as torsion, and here it does not vanish.

13.6 Contact connection

Keep $\omega^1_2 = \alpha = -\omega^2_1$ and all other $\omega^a_b = 0$ (rotation in the contact plane, generated by the fibre). Then

$$T^1 = \alpha \wedge e^2 = -\frac{1}{4} \sin \theta d\phi \wedge d\psi, \tag{13.12}$$

$$T^2 = de^2 - \alpha \wedge e^1, \tag{13.13}$$

$$T^3 = d\alpha, \tag{13.14}$$

¹⁰The *Weitzenböck* or *teleparallel* connection is the one in which the chosen frame is declared parallel everywhere. Its curvature vanishes and all geometry is carried by torsion [24].

and the torsion 3-form is

$$\sum_{a=1}^3 T^a \wedge e^a = -\frac{1}{2} \sin \theta d\theta \wedge d\phi \wedge d\psi = -4 \text{vol} \neq 0. \quad (13.15)$$

This is the explicit content of the sentence in Section 6 that the fibre piece of ω produces a contact piece of T .

In words. With a spin connection that rotates the frame in the contact plane, all three torsion components are present and their combined 3-form is four times the volume form, with a minus sign set by orientation. On this bundle the twist cannot be made to vanish.

13.7 Linking number

Two distinct Hopf fibres, stereographically projected to $\mathbb{R}^3$, were discretised (180 vertices each) and the Gauss integral evaluated¹¹. Representative values:

$$\text{Lk}(\gamma_N, \gamma_x) = 0.9997, \quad \text{Lk}(\gamma_x, \gamma_y) = 0.9999. \quad (13.16)$$

Both round to 1. This is the Hopf link of Section 5, computed rather than quoted.

In words. Computed numerically, two fibre circles pass through each other once, to four decimal places.

13.8 Stereographic connection is Beltrami

The embedding 1-form $-Y dX + X dY - W dZ + Z dW$ pulls back to α on the nose (scale factor 1). Pushed to $\mathbb{R}^3$ by stereographic projection from $W = +1$, the resulting $\mathbf{A}$ was differentiated numerically at five off-origin points. In every sample

$$\cos \angle(\mathbf{A}, \nabla \times \mathbf{A}) = 1.000000 \quad (13.17)$$

to six figures, with relative residual $\|\nabla \times \mathbf{A} - \lambda \mathbf{A}\| / \|\nabla \times \mathbf{A}\| \sim 10^{-6}$. The eigenvalue λ is position-dependent ($\lambda \approx 3.51, 2.37, 2.90, 1.38, 3.03$ on the sample). So $\mathbf{A}$ is a Beltrami field, not a constant- κ eigenfield of round S^3 . That is the expected change of frame from S^3 to $\mathbb{R}^3$.

In words. At every test point the curl of $\mathbf{A}$ points exactly along $\mathbf{A}$, which is the defining property of a Beltrami field; only the proportionality factor varies after the 3-sphere is flattened into ordinary space.

¹¹The Gauss linking integral of two closed curves is $\text{Lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_1 - \mathbf{r}_2) \cdot (d\mathbf{r}_1 \times d\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$. It is always an integer for closed curves [19, 20].

13.9 Unit quaternions give $\text{SO}(3)$

The standard conjugation matrix of a unit quaternion satisfies $RR^T = I$ and $\det R = 1$ at a numerical unit point. The product of two unit quaternions has norm 1. This is Section 3, computed.

In words. Unit quaternions really do produce proper rotations of ordinary space, and multiplying two of them gives a third.

13.10 Circular wave and the third leg

For $\mathbf{E} = (\cos \zeta, \sin \zeta, 0)$ the Hodge dual¹² in the transverse plane is $\star \mathbf{E} = (-\sin \zeta, \cos \zeta, 0)$. Setting $c\mathbf{B} = \star \mathbf{E}$ gives $\star \mathbf{E} - c\mathbf{B} = 0$. A circularly polarised wave is an eigenbivector of the spatial Hodge map. The remaining orthonormal direction is $e^3 = \alpha$, the Reeb / fibre leg. That is the UFT 439 non-singularity remark as an identity on this coframe.

In words. Rotating $\mathbf{E}$ by a quarter turn in the plane across the beam gives $c\mathbf{B}$: that is circular polarisation. The one direction left over is the fibre direction, which supplies the third leg.

13.11 What was not computed

Units of V versus $d\alpha$ in SI (UFT 457) were not matched. That remains Layer 2. A full UFT 439 Maxima tetrad-to-Christoffel dump on this coframe was not repeated in SymPy; the structure equations above are the part of that path that lives on S^3 . Mass corrections were not computed. The Cartan curvature scalar of Section 13.13 was computed for three connections on round S^3 only; time dependence was not included.

Remark (plain language). The extra circle really does not change the Hopf map. The twist 3-form on that circle is not zero; we wrote it out. Two rings in the projection really do go through each other once. The potential you get by flattening the 3-sphere to ordinary space has its curl pointing along itself. None of that required a new axiom.

¹²The *Hodge star* $\star$ maps a k -form to an $(n - k)$ -form using the metric. In a plane it turns a vector through 90° ; in three dimensions it exchanges areas and arrows, which is how the curl is written in forms [25].

13.12 Method to repeat

Files issued with the paper, each calculation in both Maxima (the AIAS house tool) and SymPy: `hopf_alignment.max / hopf_alignment_sympy.py` (Sections 13.1–13.10), `uft_details.max / uft_details_sympy.py` (Section 14) and `cartan_R.max / cartan_R_sympy.py` (Section 13.13). The Maxima files were run with Maxima 5.46.0 and reproduce the SymPy results line for line, including the linking numbers (13.16) and the Beltrami eigenvalues of Section 13.8. Linking uses 180-point polygons and the discrete Gauss integral. Beltrami derivatives use central differences with $h = 2 \times 10^{-3}$.

13.13 Cartan curvature scalar on the Hopf coframe

This subsection computes the curvature scalar in the ECE wave equation on the Hopf bundle. It uses round S^3 of radius r_0 (the coframe (13.2)–(13.4) multiplied by r_0) and the script `cartan_R_sympy.py`.

(a) *Curvature scalar of three connections.* For a connection ω^a_b on the coframe e^a define the frame contraction of the Cartan curvature 2-form (2.4),

$$R_C = \sum_{a,b} R^a_b(e_a, e_b), \quad (13.18)$$

with e_a the dual frame vectors. Then

connection	R_C	torsion	ECE field $F = A^{(0)}T$
Levi-Civita (Riemann)	$+6/r_0^2$	$T = 0$	$F = 0$
teleparallel, $\omega = 0$	0	$T \neq 0$	$F \neq 0$
contact, $\omega^1_2 = \alpha/r_0$	$-4/r_0^2$	$T \neq 0$	$F \neq 0$

The value $+6/r_0^2$ is the Einstein (Ricci) scalar; it belongs to the connection on which the ECE field vanishes.

In words. R_C adds up how much the frame turns around each pair of directions. It is $+6/r_0^2$ for the ordinary Riemannian connection (which has no torsion), zero for the teleparallel connection, and $-4/r_0^2$ for the contact connection. Three admissible connections, three different curvature numbers.

(b) *The R of the wave equation (2.10).* In ECE, R enters as the eigenvalue with which the wave operator acts on the tetrad (Evans lemma, from the tetrad postulate [2, 3]). With the ECE convention $\square = c^{-2}\partial_t^2 - \nabla^2$ (Section 2), static fields on S^3 obey $(-\nabla^2 + R)u = 0$. On 1-forms we use the vector Laplacian of the ECE vector form, $\nabla^2 u = \text{grad div } u - \text{curl curl } u$, $\text{curl } u = \star du$ (this is minus the Hodge Laplacian; the rough, or Bochner, Laplacian would

give $2/r_0^2$ instead of $4/r_0^2$ below, so the choice of operator on 1-forms is stated explicitly). The Euler coframe (13.2)–(13.4) is not an eigen-tetrad (the leg e^1 fails). Rotating it in the (e^1, e^2) plane through the fibre angle ψ , a local $SO(2)$ frame rotation that leaves the metric unchanged,

$$\tilde{e}^1 = \cos \psi e^1 + \sin \psi e^2, \quad \tilde{e}^2 = -\sin \psi e^1 + \cos \psi e^2, \quad \tilde{e}^3 = e^3, \quad (13.19)$$

gives the left-invariant¹³ Hopf coframe. SymPy and Maxima return, for every leg,

$$\text{curl } \tilde{e}^a = -\frac{2}{r_0} \tilde{e}^a, \quad -\nabla^2 \tilde{e}^a = \frac{4}{r_0^2} \tilde{e}^a. \quad (13.20)$$

Hence all three legs of the Hopf tetrad satisfy the static ECE wave equation with one common value,

$$(\square + R) \tilde{e}^a = 0 \quad \text{with} \quad R = -\frac{4}{r_0^2}, \quad (13.21)$$

which differs from the Einstein value $+6/r_0^2$ in magnitude and in sign. Each leg is Beltrami (first equation of (13.20)), so this is the tetrad form of the Hopf field of Section 10. The value coincides with R_C of the contact connection in the table, for the normalisation $\omega^1_2 = \alpha/r_0$ of Section 13.6; whether that agreement is more than a coincidence of normalisation is not decided here. The eigenvalue (13.20) is a computed identity (Layer 1); its identification with the R of ECE rests on the choice of the vector Laplacian above and is offered for checking.

In words. Turning the two horizontal arrows by the fibre angle ψ while moving round each fibre makes the frame look the same at every point of S^3 ; the metric is unchanged.

(c) *Scalar test.* A scalar on S^3 obeys $-\nabla^2 Y_l = l(l+2) r_0^{-2} Y_l$ (checked for $l = 1$). So $(\square + R)\psi = 0$ needs $R = -l(l+2)/r_0^2 \leq 0$. With R left free there is one mode for every l . The Riemannian value $R = +6/r_0^2$ admits no scalar mode at all (it would need $l(l+2) = -6$); the tetrad value (13.21) needs $l(l+2) = 4$, which has no integer solution either, and a change of radius scales both sides equally and does not help. A scalar does not share the eigenvalue of the Hopf tetrad. The tetrad itself, and spinors through the Fermion equation of Volume 2, are the candidate modes.

Remark (plain language). We asked which number R the ECE wave equation needs on the 3-sphere. The frame of the Hopf bundle answers by

¹³*Left-invariant:* unchanged when the whole of $S^3 \cong SU(2)$ is multiplied from the left by a fixed unit quaternion, so the frame looks the same at every point.

itself: all three of its arrows satisfy the equation with the same number, $R = -4/r_0^2$, not Einstein's $+6/r_0^2$. A plain scalar wave cannot use that number, so the spinning objects of Volume 2 are needed. The computation is exact; reading it as the ECE value of R depends on the choice of operator stated above, so it is offered for checking, not as a general theorem.

14 UFT details that Hopf geometry can help to close

The eight items were run in `uft_details_sympy.py`; item 5 is completed by `cartan_R_sympy.py`. Status is recorded next to each item. JSON: `sympy_results/uft_details_results.json`.

1. **Non-singular circular tetrad (UFT 439).** Computed. Gram determinant of the two rotating transverse legs is 1. The 3-volume without a third leg is 0. The Hopf 3-volume coefficient is $\sin \theta/8 \neq 0$ for $\theta \in (0, \pi)$. $T^3 = d\alpha$. Laboratory $B^{(3)}$ remains Layer 2.

14.1 UFT 439 — Circular tetrad, third leg

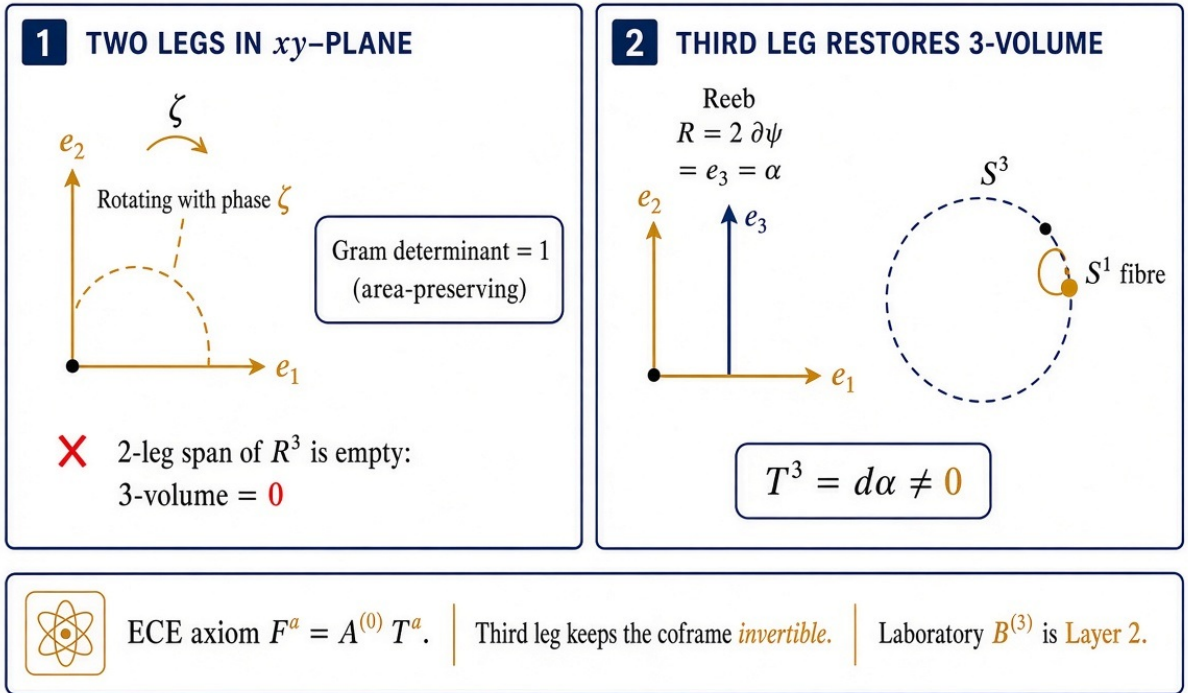


Figure 14.1. UFT 439. Two rotating legs never span $\mathbb{R}^3$. The Reeb field $R_\alpha = 2\partial_\psi$ is the third orthonormal leg. $T^3 = d\alpha$.

What the figure shows. Panel 1: two arrows rotating in a plane always span only that plane, so the volume they enclose is zero. Panel 2: adding the fibre direction (the Reeb field) as a third arrow restores a full three-dimensional frame, and its torsion $T^3 = d\alpha$ is non-zero.

2. **Potential flux V as fibre connection (UFT 454–458).** Partial. Prototypes: $[V] \sim A^{(0)}/(L_0 T_0)$ (V/m²) and $[\rho_p] \sim A^{(0)}/L_0^2$ (Vs/m³), with L_0 a length and T_0 a time scale (for example L_0/c), if $A^{(0)}$ has units Vs/m. Linked-tube energy at fixed helicity scales as $\Phi^2 R/a^2$ and rises as the minor radius shrinks. No vacuum-Maxwell concentration mechanism was obtained. SI match to UFT 457 remains Layer 2.¹⁴

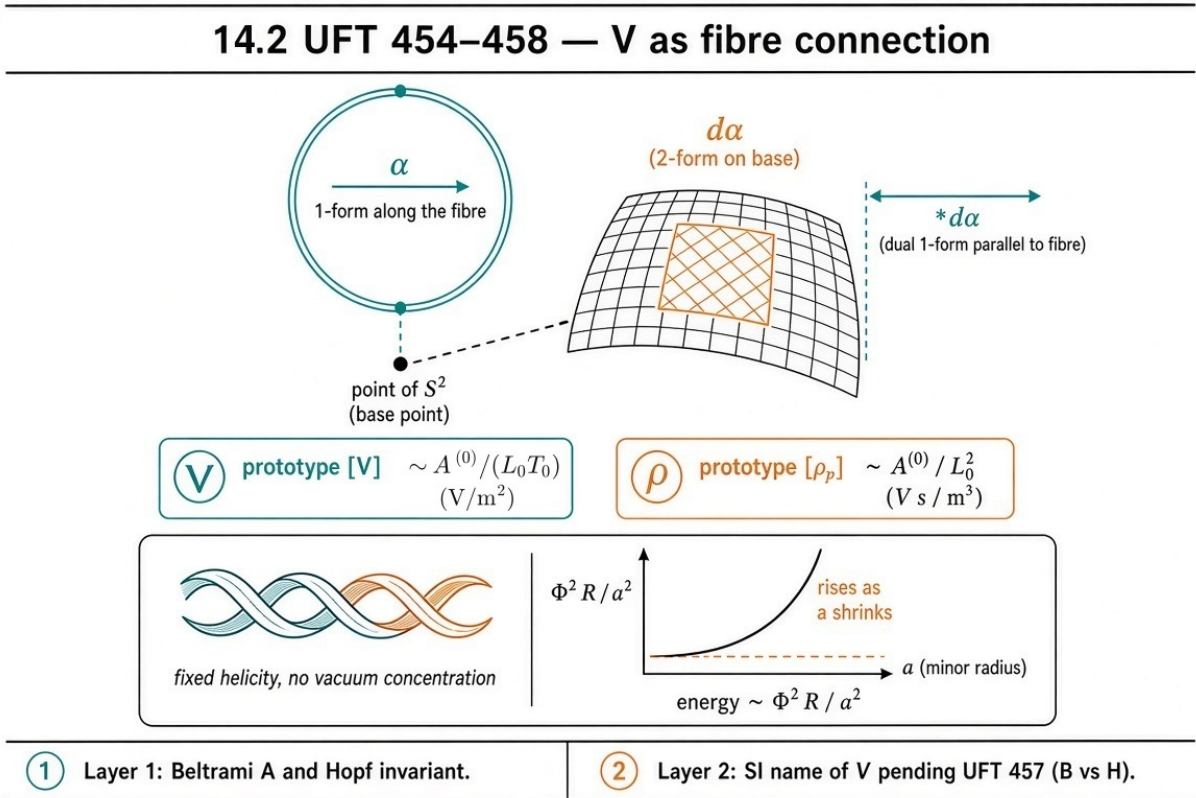


Figure 14.2. Fibre 1-form α , base 2-form $d\alpha$, and dual $\star d\alpha$ parallel to the fibre. Unit prototypes only. Energy at fixed helicity rises as the tube shrinks.

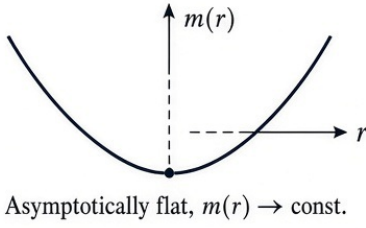
What the figure shows. The fibre 1-form α (a circle), its derivative $d\alpha$ (an area element on the base) and the dual $\star d\alpha$ (an arrow along the fibre). The boxes give unit prototypes for V and ρ_p ; the graph shows that, at fixed helicity, the energy of a linked flux tube rises as the tube is made thinner.

¹⁴The unit of V is V m⁻² [10], the unit of $\partial_t B$; in flow units this is V s m⁻² per second. Volts and amperes are dual: J is in A m⁻², V in V m⁻².

3. **Rotational field without rotational potential (UFT 449–451).** Computed. Diagonal $m(r)$ tetrad plus a fibre piece $\beta \cos \theta d\phi$ (no circulating 3-velocity). Teleparallel torsion of the spherical frame is already nonzero. The extra fibre piece contributes $T_{\text{extra}} = -\beta \sin \theta d\theta \wedge d\phi$, a radial flux. That is the swirl-from-geometry identity on a short formula.

14.3 UFT 449–451 — Swirl from geometry

1. Radial potential well $m(r)$



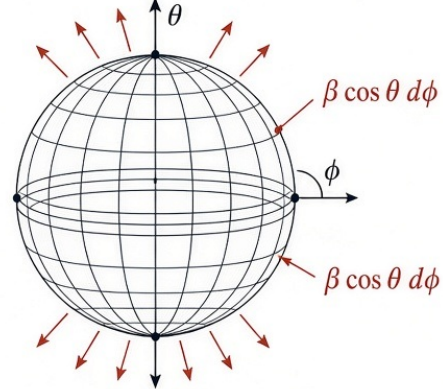
Tetrad (diagonal spherical frame)

$$\begin{Bmatrix} m^{-1/2} \\ r \\ r \sin \theta \end{Bmatrix} e_\mu^a = \text{diag}(m^{-1/2}, r, r \sin \theta)$$



no circulating 3-velocity,
no spinning source

2. Monopole / Hopf connection on the sphere



$$T_{\text{extra}} = -\beta \sin \theta d\theta \wedge d\phi$$

Radial flux out of the poles



Teleparallel torsion of the spherical frame is **already nonzero**.
Fibre piece adds a **radial rotational flux**. Geometry makes the swirl.

Figure 14.3. Diagonal $m(r)$ tetrad with no circulating 3-velocity. The Hopf/monopole piece $\beta \cos \theta d\phi$ produces a radial torsion flux.

What the figure shows. Left: a radial potential well $m(r)$ with no rotation in it. Right: adding the Hopf/monopole piece $\beta \cos \theta d\phi$ produces a torsion flux streaming out of the poles. Rotation appears from geometry, not from a rotating input.

4. **Antisymmetry constraints (Volume 1, Section 5.1).** Computed as a count and a 2-form identity. T^a is stored only as an antisymmetric 2-form. Eight field unknowns $(\Phi, \mathbf{A}, \omega_0, \boldsymbol{\omega})$, eight Maxwell-like slots, antisymmetry extra. No additional current species is introduced: the two ECE currents J and V , both consisting of fields, are the whole count.

14.4 Volume 1 §5.1 — Antisymmetry, one space

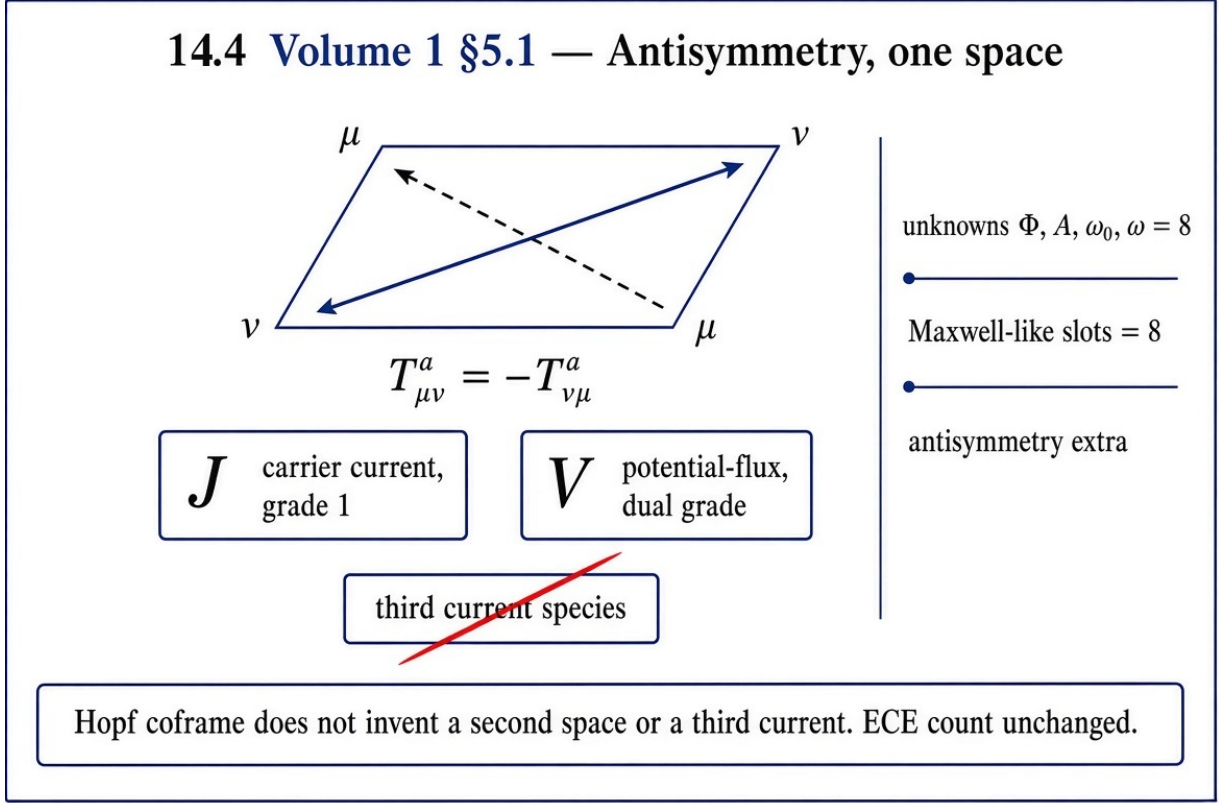


Figure 14.4. Torsion is a 2-form. One J , one V , both made of fields. The Hopf coframe does not add a further current species or a second space.

What the figure shows. Torsion components change sign when their two lower indices are swapped. Counting unknowns and equations, the system closes with the two ECE currents J and V ; no third species is needed.

5. **Wave equation** $(\square + R)\psi = 0$ on S^3 . Computed (Section 13.13). The Einstein (Ricci) scalar of the Levi-Civita connection, $+6/r_0^2$, is not the R of ECE. With the ECE convention $\square = c^{-2}\partial_t^2 - \nabla^2$, the three legs of the left-invariant Hopf tetrad satisfy the static wave equation with one common value, $R = -4/r_0^2$ (Eq. 13.21). For scalars the kernel condition then becomes $l(l+2) = 4$; no non-negative integer l works, and a change of radius does not help, so no scalar mode shares the tetrad value of R . If R is left free, $R = -l(l+2)/r_0^2$ gives a mode for every l . Spinors belong to the ECE Fermion equation of Volume 2 [3]; the Dirac spectrum $\pm(k+3/2)/r_0$ on round S^3 [30] is the natural Volume 2 example.

14.5 Volume 2 — $(\square + R)\psi = 0$ on round S^3

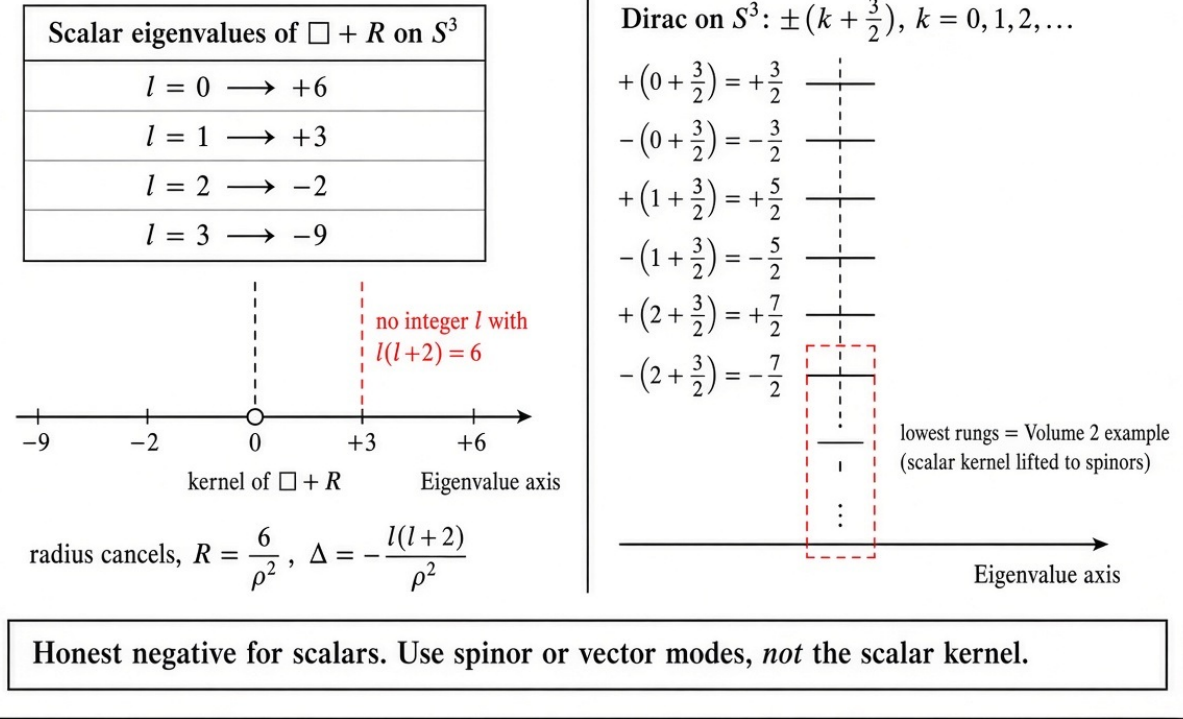


Figure 14.5. Scalar spectrum of $\square + R$ on round S^3 with the Riemannian value $R = 6/\rho^2$, which misses zero (ρ in the graphic is the radius r_0). The graphic writes the static operator as $\nabla^2 + R$, the opposite sign convention to ECE; the comparison in the ECE convention is Figure 14.5b. Dirac eigenvalues $\pm(k + 3/2)$ are the Volume 2 example of the Fermion equation.

What the figure shows. Left: the scalar eigenvalues of $\square + R$ on the 3-sphere computed with the Riemannian value $R = 6/\rho^2$; none of them is zero. Right: the eigenvalues of the Dirac operator on the same sphere, the natural example for Volume 2.

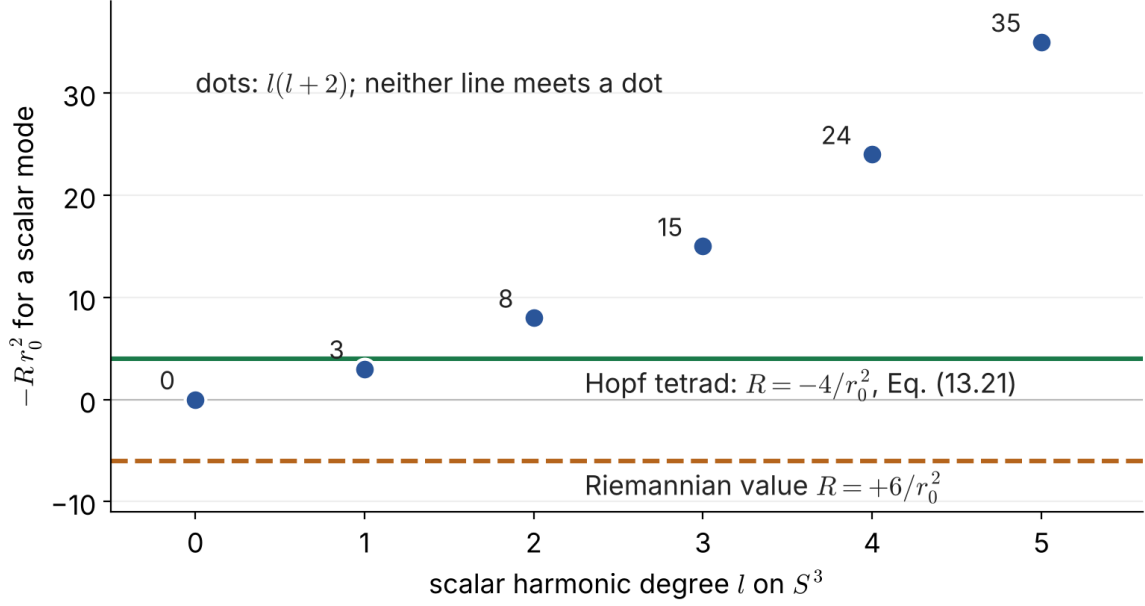


Figure 14.5b. ECE convention $\square = c^{-2}\partial_t^2 - \nabla^2$: the value of $-Rr_0^2$ needed for a scalar mode of $(\square + R)\psi = 0$ on round S^3 , by harmonic degree l (dots, $l(l+2)$). Solid: the Hopf-tetrad value $R = -4/r_0^2$, Eq. (13.21). Dashed: the Riemannian value $R = +6/r_0^2$. Neither line meets a dot; with R left free every dot is a mode.

What the figure shows. Each dot is the value of $-Rr_0^2$ that a scalar wave of degree l would need. The solid line is the value shared by the three legs of the Hopf tetrad; the dashed line is the Riemannian value, which lies below zero in this convention. No dot lies on either line.

6. **Inverse Faraday Effect as residual, not vacuum field.** Layer-1 split computed. The unique metric-compatible antisymmetric connection on this coframe was solved; its torsion vanishes (Levi-Civita). Coefficients: $\omega^1_2 = e^3 - 2 \cot \theta e^2$, $\omega^1_3 = e^2$, $\omega^2_3 = -e^1$. Contact and teleparallel choices have $T \neq 0$. ECE $F = A^{(0)}T$ is therefore zero in the LC vacuum sector. Laboratory IFE is not computed.

14.6 Residual split — LC vacuum vs contact torsion

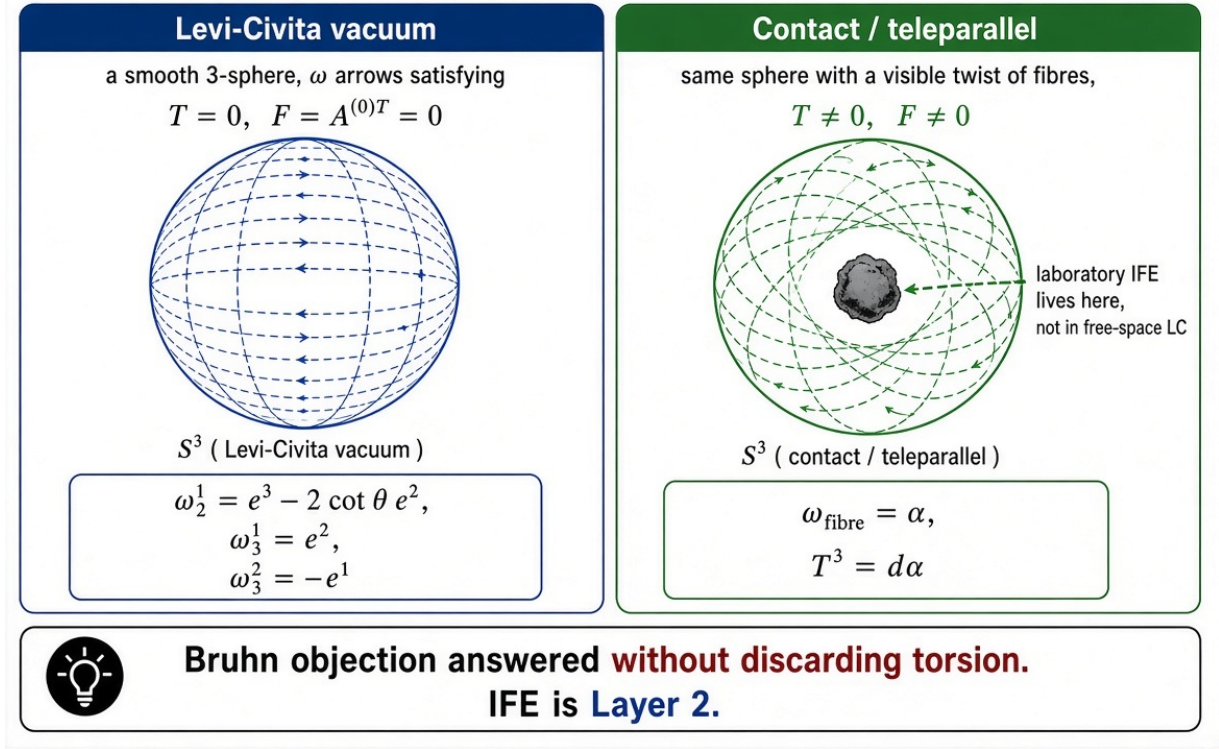
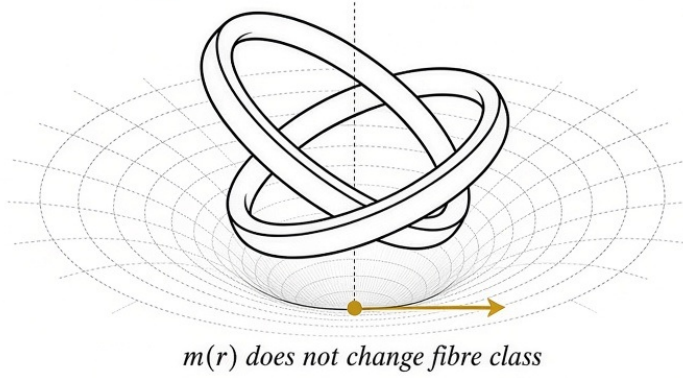


Figure 14.6. Levi-Civita sector: $T = 0$ hence $F = 0$. Contact/teleparallel sector: $T \neq 0$. Laboratory IFE is not a free-space field.

What the figure shows. Left: with the Levi-Civita connection the torsion vanishes, so the ECE field $F = A^{(0)T}$ is zero. Right: with the contact (or teleparallel) connection the fibres are visibly twisted, torsion is present, and any laboratory Inverse Faraday Effect would belong here, with matter present.

7. **m-theory central force plus Hopf linking.** Integer computed; mass number not computed. $|\int \alpha \wedge d\alpha| / (4\pi^2) = 1$. A radial $m(r)$ factor does not change the fibre class. The ansatz $\Delta m/m = c_{\text{mass}} H$ only names c_{mass} . It is not a neutron-mass derivation.

14.7 Hopf invariant is the only integer



$$\frac{(m_n - m_p)}{m_p}$$

not computed.

$$\text{--- } \mathbf{H} = 1 \text{ ---}$$

$$\left| \int \frac{\alpha \wedge d\alpha}{4\pi^2} \right| = 1$$



Ansatz $\Delta m/m = c_{mass}$ H only names c_{mass}

Mass number stays off the axiom list.

Figure 14.7. The bundle supplies the integer $H = 1$. A radial $m(r)$ factor does not change the fibre class. No neutron mass is computed.

What the figure shows. Two linked rings carry the Hopf invariant $H = 1$. Scaling space radially by $m(r)$ does not change that integer. The crossed-out box records that no neutron–proton mass difference is derived.

8. **Clifford form (Goel / Section 12).** Grades computed. α is grade 1, $d\alpha$ grade 2, $\alpha \wedge d\alpha$ grade 3, $\star d\alpha$ grade 1 and parallel to the fibre. That is Beltrami in $Cl(3)$. Matching the dual grade to V in UFT 457 remains Layer 2.

14.8 Clifford grades of the Hopf coframe

Grade 0	scalar 	1 scalar
Grade 1	vector 	$\alpha \sim e_3$ vector and also $*d\alpha \parallel \alpha$
Grade 2	bivector 	$d\alpha = -2 e_1 \wedge e_2$ bivector
Grade 3	pseudoscalar 	$\alpha \wedge d\alpha = -2 I$ pseudoscalar

Hodge star
($*$: grade 2 $\rightarrow$ grade 1)
along the fibre





Beltrami in $Cl(3)$:
 $*d\alpha \parallel \alpha$

Goel / UFT 456 bookkeeping. Matching the dual grade to V is Layer 2. ECE axioms unchanged.

Figure 14.8. Clifford grades of the Hopf coframe. $*d\alpha$ is parallel to α : Beltrami in $Cl(3)$. V remains Layer 2.

What the figure shows. The grade of each object on the Hopf coframe: α is a vector (grade 1), $d\alpha$ a bivector (grade 2), $\alpha \wedge d\alpha$ a pseudoscalar (grade 3). The Hodge star turns $d\alpha$ back into a vector parallel to α : the Beltrami property in Clifford language.

Remark (plain language). Six of the eight items now have a Layer-1 identity (one of them a negative: no scalar kernel of $\square + R$ on round S^3 , now shown with the ECE value of R). Two items (V units, mass number) stay Layer 2 on purpose. The method is the same: compute, keep what holds, drop what does not.

15 What is not claimed

- LLM embedding spaces are not Hopf bundles.
- An unverified agent manuscript on forced Navier–Stokes is not a measurement of linking.
- “Cold current” is not used as a primary name. The object is V , as in UFT 454–458.

- Chamber dimensions, device couplings, cascade rates, and enablement protocols are omitted.
- No identification of the Hopf integer with the mass ratio $\delta = (m_n - m_p)/m_p$ is made.
- Curvature is not claimed to be a current or a source. Currents consist of fields.
- The value $R = -4/r_0^2$ is computed for the round Hopf coframe only, with the operator stated in Section 13.13; it is not claimed for other geometries.

16 Dictionary for Volume 1

Hopf / contact object	ECE object (Volume 1 name)	Layer
$S^3 \cong SU(2)$	spinor / tetrad bundle	1
$S^2 \cong \mathbb{CP}^1$	Bloch sphere; ray space	1
fibre S^1	$U(1)$ phase; third tetrad leg	1
contact form α	fibre piece of ω	1
$\alpha \wedge d\alpha$	contact piece of T^a	1
Reeb / Hopf field	model Beltrami field $T = \kappa q$	1
$H(\pi) = 1$	unit of helicity	1
non-singular circular tetrad	UFT 439 third component	1
Rañada hopfion	Maxwell section, $\delta\omega = 0$	1
left-invariant Hopf tetrad	eigen-tetrad, $-\nabla^2 \tilde{e}^a = (4/r_0^2) \tilde{e}^a$	1
same, read as $(\square + R)q = 0$	ECE $R = -4/r_0^2$ (vector Laplacian; offered for checking)	2
first Bianchi identity	homogeneous field equation; currents made of fields	1
fibre connection $\rightarrow V$	proposed prototype of potential flux	2
IFE from third leg + matter	residual, not free-space $B^{(3)}$	2
$\Delta Lk \rightarrow \Delta m$	neutron-excess hypothesis	2
$Cl(0, 2) \cong \mathbb{H}$	Hopf total space as algebra	1
$F = \mathbf{E} + Ic\mathbf{B}$	STA Faraday bivector	1
$\nabla F = J$	STA Maxwell rewrite	1
dual grade of $\mathcal{J}$	proposed prototype of V	2

17 Status of supporting calculations

The following calculations support the text; their status is as follows.

1. **Maxima cross-check of the coframe.** Done. Files `hopf_alignment.max` and `uft_details.max` reproduce $d\alpha = -2e^1 \wedge e^2$, the Reeb field, teleparallel T , the Levi-Civita solution with $T = 0$, and $\int \alpha \wedge d\alpha = -4\pi^2$.
2. **SI units of $d\alpha$ versus V .** Dimensions consistent; identification open. With $A^{(0)}$ in V s m^{-1} , a length L_0 and a time scale T_0 , the prototypes $[V] \sim A^{(0)}/(L_0 T_0)$ (V m^{-2}) and $[\rho_p] \sim A^{(0)}/L_0^2$ (V s m^{-3}) carry the units of UFT 456. Whether $\star d\alpha$ is the physical V remains Layer 2.
3. **$m(r)$ on the Hopf coframe.** Done for a diagonal spatial tetrad plus the fibre piece $\beta \cos \theta d\phi$. The extra torsion is $-\beta \sin \theta d\theta \wedge d\phi$. A conformal factor does not change the integer fibre class.
4. **Antisymmetry component by component.** Done for the teleparallel Hopf coframe. The only nonzero coordinate torsion components of $T^\lambda_{\theta\phi}$ are

$$T^\theta_{\theta\phi} = 0, \quad T^\phi_{\theta\phi} = \cot \theta, \quad T^\psi_{\theta\phi} = -1/\sin \theta, \quad (17.1)$$

with $T^\lambda_{\phi\theta} = -T^\lambda_{\theta\phi}$. No additional current species appears.

In words. Each torsion component changes sign when its two lower indices are swapped, which is what a 2-form means. Nothing in the list behaves like an additional current.

18 Laboratory tests not computed in this paper

The following are experimental proposals, not results.

1. Helicity of two linked Beltrami or hopfion configurations before and after reconnection. A Monte Carlo integral of $\mathbf{A} \cdot (\nabla \times \mathbf{A})$ in a ball of radius 2 in the stereographic chart is cutoff- and gauge-dependent (sample value ≈ 34 at $N = 4000$). The integer $H = 1$ lives on S^3 , not in that chart.
2. Inverse Faraday protocol: longitudinal response only with matter present; null in the Levi-Civita vacuum sector $\delta\omega = 0$.
3. Discriminator for V versus J : the two-plate test outside a Faraday cage published in the UFT 454–458 series. No coil recipe is added here.

19 Conclusions

The Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$ is a compact total space on which the ECE axioms of Volume 1 can be written globally. The orthonormal coframe of the round metric has contact form $\alpha = e^3$ with $\alpha \wedge d\alpha = -2 \text{ vol}$. The third tetrad leg is the Reeb field. Teleparallel and contact connections have $T \neq 0$; the unique metric-compatible antisymmetric connection on the same coframe has $T = 0$. ECE $F = A^{(0)}T$ therefore vanishes in the Levi-Civita vacuum sector and survives in the contact sector. Two projected fibres have linking number 1. The stereographic image of α is Beltrami. The R of the ECE wave equation is not the Einstein scalar: the Levi-Civita value $6/r_0^2$ belongs to the connection on which $F = 0$. The left-invariant Hopf tetrad is an eigen-tetrad of the wave operator, so it satisfies $(\square + R)q = 0$ with one common value $R = -4/r_0^2$ for all three legs (ECE convention for $\square$; vector Laplacian on 1-forms); scalar eigenfunctions do not meet the corresponding kernel condition $l(l+2) = 4$, and spinors are left to the Fermion equation of Volume 2. Currents, ordinary and homogeneous, are made of the geometric fields; the Hopf coframe adds no current species. The only integer the bundle supplies is the Hopf invariant 1. Identifications of that integer with a mass number, and of $\star d\alpha$ with the SI object V of UFT 457, remain Layer 2.

Computer-algebra files (SymPy and Maxima) that generate the identities are issued with the paper.

20 Definitions

Terms are listed in the order a reader first meets them.

Term	Meaning in this paper
Tetrad q^a	local coframe: four (here three) reference arrows at each point; the potential prototype of ECE
Spin connection ω	rule that says how the tetrad rotates from point to point
Torsion T^a	Dq^a : the twist or shear of the frame; the field prototype of ECE
Curvature R^a_b	$D\omega$: the rotation of the frame around a small loop
First Bianchi identity	$DT^a = R^a_b \wedge q^b$; the homogeneous field equation of ECE
Currents	ordinary (J) and homogeneous (V); both consist of fields, not external sources

Term	Meaning in this paper
Tetrad postulate	$D_\mu q^a{}_\nu = 0$: the complete covariant derivative of the tetrad vanishes
Evans lemma / ECE wave equation	$(\square + R)q^a{}_\mu = 0$, obtained from the tetrad postulate
Curvature scalar R (ECE)	scalar in the ECE wave equation; not Einstein's $g^{\mu\nu}R_{\mu\nu}$; no sign restriction
Fibre bundle	space built from a base with a copy of a fibre (here a circle) over each point
Hopf fibration	$S^1 \rightarrow S^3 \rightarrow S^2$, Chern class $c_1 = 1$
Chern class c_1	integer that counts the twisting of a circle bundle
Double cover	two-to-one map $SU(2) \rightarrow SO(3)$; the reason spinors need 4π
Stereographic projection	projection of a sphere minus one point onto flat space; circles stay circles
Contact form α	connection 1-form along the Hopf fibre; $\alpha \wedge d\alpha \neq 0$
Reeb field R_α	vector field along the fibre with $\alpha(R_\alpha) = 1$, $\iota_{R_\alpha} d\alpha = 0$
Linking number	number of times one closed curve winds through another
Hopf invariant	$\int \alpha \wedge d\alpha$ (normalised): the linking number of two fibres, equal to 1
Helicity	$\int \mathbf{A} \cdot \mathbf{B} d^3x$: field-line linking in vector form
Beltrami condition	curl parallel to the field: $\nabla \times \mathbf{u} = \lambda \mathbf{u}$; in ECE $T = \kappa q$
Hopfion	field configuration whose lines are Hopf links (Rañada solution)
Residual $\delta\omega$	part of the spin connection beyond Levi-Civita; set to 0 in free space
Levi-Civita connection	the unique torsion-free metric connection of Riemann geometry
Teleparallel connection	connection with zero curvature; all geometry in torsion
Contact connection	connection $\omega^1{}_2 = \alpha$ rotating the frame in the contact plane
Left-invariant coframe	frame that looks the same at every point of $S^3 \cong SU(2)$
Hodge star $\star$	map from k -forms to $(n - k)$ -forms using the metric

Term	Meaning in this paper
Bloch sphere	S^2 of physical states of a two-level quantum system
Spinor	object that changes sign under a 2π rotation and returns after 4π
ECE Fermion equation	spinor equation of Volume 2, equivalent to the Dirac equation
Geometric product ab	$a \cdot b + a \wedge b$
Grade	scalar (0), vector (1), bivector (2), trivector / pseudoscalar (3), $\dots$
Pseudoscalar I	highest-grade unit element; $I^2 = -1$ in 3-space
STA	$Cl(1, 3)$, spacetime algebra
$\mathcal{J}$	proposed multivector source
Potential flux V	homogeneous current density of UFT 454–458; not ohmic J
$B^{(3)}$	longitudinal magnetic field proposed by Evans; not used as a Layer-1 field here
Inverse Faraday Effect	magnetisation of matter by circularly polarised light (Layer 2 here)
Layer 1	topology + Cartan geometry + ECE axioms
Layer 2	proposed physical identifications awaiting calculation or measurement

21 Symbols

SI units are given where they are fixed; “1” means dimensionless. Indices: a, b, c are frame (tangent-space) indices; μ, ν, λ are coordinate indices.

Symbol	Name	SI unit	Role / where
<i>Cartan geometry and ECE</i>			
$q^a{}_\mu, q^a$	tetrad	1	potential prototype (Sec. 2)
$\omega^a{}_b$	spin connection	m^{-1}	always present
$\Gamma^\lambda{}_{\mu\nu}$	affine connection	m^{-1}	tetrad postulate
$g_{\mu\nu}$	metric	1	ds^2
T^a	torsion 2-form	m^{-1}	field prototype
$R^a{}_b$	Cartan curvature 2-form	m^{-2}	enters $DT = R \wedge q$

Symbol	Name	SI unit	Role / where
D	exterior covariant derivative	m^{-1}	Eqs. (2.3)–(2.9)
$A^{(0)}$	ECE scale factor	V s m^{-1}	not fixed here
A^a, F^a	ECE potential and field	$\text{V s m}^{-1};$ V s m^{-2}	Eqs. (2.1)–(2.2)
G, Q	gravitational field and potential	–	Eq. (2.8)
F_{fd}, v	fluid vorticity and velocity	$\text{s}^{-1}; \text{m s}^{-1}$	Eq. (2.9)
R	ECE curvature scalar	m^{-2}	$(\square + R)q = 0$; no sign restriction
R_C	frame-contracted Cartan curvature	m^{-2}	Eq. (13.18)
$\square$	d'Alembert operator $c^{-2}\partial_t^2 - \nabla^2$	m^{-2}	Eq. (2.10)
∇^2	vector Laplacian	m^{-2}	Sec. 13.13
<i>Hopf geometry</i>			
z_1, z_2	complex coordinates on S^3	1	Eq. (3.1)
$x_1, \dots, x_4$	real coordinates; X, Y, Z, W embedding	1	Eq. (13.5), Sec. 13.8
π	Hopf map	–	Eq. (4.1)
θ, ϕ, ψ	Euler / Hopf angles	rad	Sec. 13
$e^a, \tilde{e}^a$	Euler and left-invariant coframes	m (radius r_0)	Eqs. (13.2)– (13.4), (13.19)
e_a	dual frame vectors	m^{-1}	Eq. (13.18)
α	contact form	m	fibre connection
R_α	Reeb field	m^{-1}	$= 2\partial_\psi$ on unit S^3
ω_{S^2}	area form of the base	1	Def. 5.2
c_1	first Chern class	1 (integer)	Prop. 4.1
$H(\pi)$	Hopf invariant	1 (integer)	Eq. (5.1)
Lk	linking number	1 (integer)	Eq. (13.16)
$\gamma_N, \gamma_x, \gamma_y$	sample fibres	–	Eq. (13.16)
ϑ, ζ	fibre phase; wave phase	rad	Eq. (13.6), Sec. 13.10
r_0	radius of S^3	m	Sec. 13.13 (ρ in Fig. 14.5)
l, Y_l	harmonic degree; scalar harmonic	1	Sec. 13.13
k	Dirac level	1	$\pm(k + 3/2)/r_0$
λ	curl / Laplacian eigenvalue	m^{-1} or m^{-2}	Sec. 13.8, Sec. 13.13

Symbol	Name	SI unit	Role / where
<i>Fields and currents (vector form)</i>			
$\mathbf{E}, \mathbf{D}$	electric field; displacement	$\text{V m}^{-1};$ C m^{-2}	Eqs. (8.1)–(8.4)
$\mathbf{B}, \mathbf{H}$	magnetic induction; magnetic field	$\text{T}; \text{A m}^{-1}$	Eqs. (8.1)–(8.4)
$\mathbf{A}$	vector potential (stereographic)	V s m^{-1}	Def. 5.0
$\mathbf{J}$	electric current density	A m^{-2}	Eq. (8.4)
$\mathbf{V}$	potential-flux area density	V m^{-2} [10]	Eq. (8.2)
ρ_e, ρ_p	charge density; potential density	$\text{C m}^{-3};$ V s m^{-3}	Eqs. (8.1), (8.3)
I_H	potential current	–	UFT 454–458
$\kappa, \kappa^{(\Lambda)}$	wave-number vectors; Beltrami eigenvalue	m^{-1}	Sec. 8, Sec. 10
$\mathcal{H}$	magnetic helicity	$\text{T}^2 \text{m}^4$ equiv.	Eq. (5.2)
Φ, R, a	flux; major and minor tube radii	$\text{Wb}; \text{m}; \text{m}$	Sec. 14 item 2
$m(r), \beta$	m-theory metric function; fibre coefficient	1	Sec. 9, Sec. 14 item 3
c_{mass}	named (not derived) mass coefficient	1	Sec. 14 item 7
$\delta\omega$	residual spin connection	m^{-1}	Sec. 7
<i>Clifford algebra</i>			
$Cl(p, q)$	Clifford algebra of signature (p, q)	–	Eq. (12.2)
I	unit pseudoscalar	1	$I^2 = -1$ in 3-space
F	STA Faraday bivector	V s m^{-2} equiv.	$\mathbf{E} + Ic\mathbf{B}$
$\mathcal{J}$	multivector source	–	Prop. 12.1
$\star, \iota$	Hodge star; interior product	–	Sec. 13
c	speed of light	m s^{-1}	–

Note: R also denotes the major tube radius in Sec. 14 item 2, and H is used both for the Hopf invariant and, in bold, for the magnetic field $\mathbf{H}$; the meaning is fixed by context.

22 Acronyms and abbreviations

Acronym	Meaning
AIAS	Alpha Institute for Advanced Studies (www.aias.us)
CAS	computer algebra system (here SymPy and Maxima)
$\mathbb{CP}^1$	complex projective line, $\cong S^2$
ECE	Einstein–Cartan–Evans (unified field theory)
GA	geometric algebra (Clifford algebra)
GCUFT	Generally Covariant Unified Field Theory (book series [1])
GTG	Gauge Theory Gravity (Lasenby, Doran and Gull [14])
IFE	Inverse Faraday Effect
JSON	JavaScript Object Notation (data file format of the CAS output)
LC	Levi-Civita (connection)
LLM	large language model
RTI	Resonant Technologies Inc., Calgary
SI	International System of Units
SO(3)	group of rotations of 3-space
Spin(3)	spin group, $\cong SU(2)$
STA	spacetime algebra, $Cl(1, 3)$
SU(2)	special unitary group of 2×2 matrices, $\cong S^3$
U(1)	circle group of unit complex numbers (phases)
UCRC-CG	Unified Classical Resonance Cosmology in Cartan Geometry [40]
UFT	Unified Field Theory: the numbered AIAS paper series
UPITEC	Unified Physics Institute of Technology (www.upitec.org)
Vol. 1, Vol. 2	H. Eckardt, ECE textbook, Volumes 1 and 2 [2, 3]

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(October 2026).

Note on preparation (C. M. Wulf). My part of this work was done through prompts to AI language-model assistants. I acted mainly as manager and go-between: I set the questions, passed H. Eckardt’s comments to the AI tools, and checked that each comment was answered. The detailed mathematics and geometry are beyond what I could work through alone; I am learning them as the work goes on. The computed identities were checked with the computer-algebra files listed in Section 13.12, and the physics rests on the calculation path and review of H. Eckardt.

Figure credits and notes. Figures are teaching graphics; none is laboratory data. Figures 4.3 and 4.4 are after Niles Johnson [34] and N. García Belmonte (PhiloGB) [35]; Figures 3.1, 11.1 and 12.2 are after standard teaching graphics as stated in their captions. For print, the black backgrounds and bars of Figures 3.1, 4.3 and 4.4 were removed; no content was changed. Figure 2.3 is after Capozziello, De Falco and Ferrara [41]; its labels are used here only for the geometric distinction between curvature and torsion. Figures 7.1, 7.2 and 14.5b were drawn for this paper.

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