Finding Beltrami fields by an eigenvalue method

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Abstract

A numerical method for computing Beltrami fields is presented. This method was suggested by the similarity between the Beltrami condition and eigenvalue problems. A Beltrami field of interest is expanded into a series of basis functions, similarly to how eigenvalue problems are handled in solid-state physics. The result is a special eigenvalue equation. This method is worked out for a plane wave basis. Solutions can be computed numerically.

Keywords: Beltrami fields, eigenvalue equations, numerical methods.

1 Introduction

The Einstein-Cartan-Evans (ECE) theory [1, 2], which is based on Cartan geometry, completed Einstein's general relativity, and has continued to advance many areas of physics. Beltrami fields have been shown to be valid solutions of the ECE field equations, which are Maxwell-like equations that have been investigated in the UFT series of ECE papers (available on www.aias.us). The characteristics of Beltrami fields have also been described in a number of ECE papers [3, 4], in which the Beltrami solutions were analytical functions for special cases of application. In order to use Beltrami fields in broader application areas, a general solution method is desirable. Finite Element methods have been developed [6, 7], but they are based on highly specialized mathematical methods that are not so easy to understand.

As an alternative, the method used in this paper is based on standard procedures of computational quantum chemistry and solid-state physics [5]. These procedures lead to eigenvalue equations, and have already been used for computing elementary particles in ECE theory [3]. In this paper, a practical computational method for Beltrami fields is developed in a comparable way, and it can be implemented using a numerical package or a computer algebra system.

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2 Elaboration of the method

A Beltrami field $\mathbf{F}$ is defined by the relation

$$\boldsymbol{\nabla} \times \mathbf{F} = \kappa \mathbf{F}, \tag{1}$$

where κ is a constant. In general, κ could be allowed to be a scalar function, but here we restrict ourselves to the constant case. Eq. (1) can be written in the form

$$\mathbb{A}\mathbf{F} - \kappa \mathbf{F} = \mathbf{0}, \tag{2}$$

where the operator $\mathbb{A}$ means $\boldsymbol{\nabla}\times$.

The solution of this equation can be developed in the following way. We start by considering the three vector components of $\mathbf{F}$ separately:

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}. \tag{3}$$

We then expand the components F_j by a set of basis functions $\phi_i(\mathbf{r})$:

$$F_j(\mathbf{r}) = \sum_{i=1}^{N} c_{ij} \phi_i(\mathbf{r}) \tag{4}$$

with development coefficients c_{ij}. The coefficients have two indices: $i = 1 \ldots N$ is the index of functional expansion, and $j = 1, 2, 3$ denotes the components of $\mathbf{F}$. The aim is to find the coefficients c_{ij} and eigenvalues κ.

The curl operator in cartesian coordinates is known to be

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{bmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{bmatrix}, \tag{5}$$

with partial derivatives $\partial_1 = \frac{\partial}{\partial X}$ etc. Inserting Eq. (4) into the components of (2) gives $3N$ equations

$$\sum_{i=1}^{N} (c_{i3}\ \partial_2 \phi_i - c_{i2}\ \partial_3 \phi_i) - \sum_{i=1}^{N} \kappa\ c_{i1}\ \phi_i = 0, \tag{6}$$

$$\sum_{i=1}^{N} (c_{i1}\ \partial_3 \phi_i - c_{i3}\ \partial_1 \phi_i) - \sum_{i=1}^{N} \kappa\ c_{i2}\ \phi_i = 0, \tag{7}$$

$$\sum_{i=1}^{N} (c_{i2}\ \partial_1 \phi_i - c_{i1}\ \partial_2 \phi_i) - \sum_{i=1}^{N} \kappa\ c_{i3}\ \phi_i = 0. \tag{8}$$

In all equations, the functions ϕ_i appear without regard to the component index, because all components are developed by the same set of functions.

To determine the coefficients c_{ij}, we apply a variation principle that is used, for example, to solve the Schrödinger equation in quantum mechanics [5]. To

minimize the left sides of Eqs. (6-8), we expand the functions into the complex plane and multiply them by an additional conjugate sum of type $(c_{ij}\phi_i)^*$:

$$\sum_{j=1}^{N}(c_{j1}\ \phi_j)^*\left(\sum_{i=1}^{N}(c_{i3}\ \partial_2\phi_i - c_{i2}\ \partial_3\phi_i) - \sum_{i=1}^{N}\kappa\ c_{i1}\ \phi_i\right) = 0, \tag{9}$$

$$\sum_{j=1}^{N}(c_{j2}\ \phi_j)^*\left(\sum_{i=1}^{N}(c_{i1}\ \partial_3\phi_i - c_{i3}\ \partial_1\phi_i) - \sum_{i=1}^{N}\kappa\ c_{i2}\ \phi_i\right) = 0, \tag{10}$$

$$\sum_{j=1}^{N}(c_{j3}\ \phi_j)^*\left(\sum_{i=1}^{N}(c_{i2}\ \partial_1\phi_i - c_{i1}\ \partial_2\phi_i) - \sum_{i=1}^{N}\kappa\ c_{i3}\ \phi_i\right) = 0. \tag{11}$$

After multiplying out the sums, we obtain the double sums

$$\sum_{i,j=1}^{N} c_{j1}^*\ \phi_j^*\ (c_{i3}\ \partial_2\phi_i - c_{i2}\ \partial_3\phi_i) - \sum_{i,j=1}^{N}\kappa\ c_{j1}^*\ \phi_j^* c_{i1}\ \phi_i = 0, \tag{12}$$

$$\sum_{i,j=1}^{N} c_{j2}^*\ \phi_j^*\ (c_{i1}\ \partial_3\phi_i - c_{i3}\ \partial_1\phi_i) - \sum_{i,j=1}^{N}\kappa\ c_{j2}^*\ \phi_j^* c_{i2}\ \phi_i = 0, \tag{13}$$

$$\sum_{i,j=1}^{N} c_{j3}^*\ \phi_j^*\ (c_{i2}\ \partial_1\phi_i - c_{i1}\ \partial_2\phi_i) - \sum_{i,j=1}^{N}\kappa\ c_{j3}^*\ \phi_j^* c_{i3}\ \phi_i = 0. \tag{14}$$

Rearranging the ϕ_j^*'s gives

$$\sum_{i,j=1}^{N} c_{j1}^*\left(c_{i3}\ \phi_j^*\partial_2\phi_i - c_{i2}\ \phi_j^*\partial_3\phi_i\right) - \sum_{i,j=1}^{N}\kappa\ c_{j1}^*\ c_{i1}\ \phi_j^*\phi_i = 0, \tag{15}$$

$$\sum_{i,j=1}^{N} c_{j2}^*\left(c_{i1}\ \phi_j^*\partial_3\phi_i - c_{i3}\ \phi_j^*\partial_1\phi_i\right) - \sum_{i,j=1}^{N}\kappa\ c_{j2}^*\ c_{i2}\ \phi_j^*\phi_i = 0, \tag{16}$$

$$\sum_{i,j=1}^{N} c_{j3}^*\left(c_{i2}\ \phi_j^*\partial_1\phi_i - c_{i1}\ \phi_j^*\partial_2\phi_i\right) - \sum_{i,j=1}^{N}\kappa\ c_{j3}^*\ c_{i3}\ \phi_j^*\phi_i = 0. \tag{17}$$

Applying integration over the whole space gives

$$\sum_{i,j=1}^{N} c_{j1}^*\left(c_{i3}\int\phi_j^*\partial_2\phi_i d\tau - c_{i2}\int\phi_j^*\partial_3\phi_i d\tau\right) - \sum_{i,j=1}^{N}\kappa\ c_{j1}^*\ c_{i1}\int\phi_j^*\phi_i d\tau = 0, \tag{18}$$

$$\sum_{i,j=1}^{N} c_{j2}^*\left(c_{i1}\int\phi_j^*\partial_3\phi_i d\tau - c_{i3}\int\phi_j^*\partial_1\phi_i d\tau\right) - \sum_{i,j=1}^{N}\kappa\ c_{j2}^*\ c_{i2}\int\phi_j^*\phi_i d\tau = 0, \tag{19}$$

$$\sum_{i,j=1}^{N} c_{j3}^*\left(c_{i2}\int\phi_j^*\partial_1\phi_i d\tau - c_{i1}\int\phi_j^*\partial_2\phi_i d\tau\right) - \sum_{i,j=1}^{N}\kappa\ c_{j3}^*\ c_{i3}\int\phi_j^*\phi_i d\tau = 0. \tag{20}$$

The integrals depend only on the basis functions and their derivatives. To obtain solutions for the coefficients c_{ik}, we first apply the variational principle, thus

deriving the three equations for c_{j1}^*, c_{j2}^* and c_{j3}^*. These derived equations then have to be set to zero, which is already the case here from the definition of the equations. Applying this procedure gives us

$$\sum_{i,j=1}^{N} \left(c_{i3} \int \phi_j^* \partial_2 \phi_i d\tau - c_{i2} \int \phi_j^* \partial_3 \phi_i d\tau \right) - \sum_{i,j=1}^{N} \kappa \; c_{i1} \int \phi_j^* \phi_i d\tau = 0, \quad (21)$$

$$\sum_{i,j=1}^{N} \left(c_{i1} \int \phi_j^* \partial_3 \phi_i d\tau - c_{i3} \int \phi_j^* \partial_1 \phi_i d\tau \right) - \sum_{i,j=1}^{N} \kappa \; c_{i2} \int \phi_j^* \phi_i d\tau = 0, \quad (22)$$

$$\sum_{i,j=1}^{N} \left(c_{i2} \int \phi_j^* \partial_1 \phi_i d\tau - c_{i1} \int \phi_j^* \partial_2 \phi_i d\tau \right) - \sum_{i,j=1}^{N} \kappa \; c_{i3} \int \phi_j^* \phi_i d\tau = 0. \quad (23)$$

All three equations can be combined into one equation of the form

$$\sum_{i,j=1}^{N} \left(c_{ik_3} \int \phi_j^* \partial_{k_2} \phi_i d\tau - c_{ik_2} \int \phi_j^* \partial_{k_3} \phi_i d\tau \right) - \sum_{i,j=1}^{N} \kappa \; c_{ik_1} \int \phi_j^* \phi_i d\tau = 0, \quad (24)$$

where the indices (k_1, k_2, k_3) are a cyclic permutation of $(1, 2, 3)$.

The integrals only depend on the basis functions and represent structure constants. We denote these by

$$B(j,i) = \int \phi_j^* \phi_i d\tau \quad (25)$$

and

$$D(j,i,k) = \int \phi_j^* \partial_k \phi_i d\tau. \quad (26)$$

Eq. (24) can then be written in the form

$$\sum_{i,j=1}^{N} \left(c_{ik_3} D(j,i,k_2) - c_{ik_2} D(j,i,k_3) \right) - \sum_{i,j=1}^{N} \kappa \; c_{ik_1} B(j,i) = 0. \quad (27)$$

B represents the overlap integral of non-orthogonal basis functions. If the basis set is orthogonal, we have

$$B(j,i) = \int \phi_j^* \phi_i d\tau = \delta_{ji}, \quad (28)$$

so that the double sum for B reduces to a single sum:

$$\sum_{i,j=1}^{N} \left(c_{ik_3} D(j,i,k_2) - c_{ik_2} D(j,i,k_3) \right) - \sum_{i=1}^{N} \kappa \; c_{ik_1} = 0. \quad (29)$$

To further specify the eigenvalue equation, we choose a plane wave expansion for the basis functions ϕ_i:

$$\phi_{\mathbf{G}} = \frac{1}{\sqrt{N^3}} \sum_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{x}}, \quad (30)$$

where i is the imaginary unit, not to be confused with an index. The vector $\mathbf{G}$ is now a three-dimensional indexing mechanism for the ϕ's. The $\mathbf{G}$'s are wave vectors[1], representing three-dimensional expansion coefficients, and the phase

[1] In solid state physics, these are vectors of the reciprocal lattice.

factor is

$$\mathbf{G} \cdot \mathbf{x} = G_1 x_1 + G_2 x_2 + G_3 x_3. \tag{31}$$

As is well known, plane waves are orthogonal functions, so in this case:

$$\begin{aligned} B(\mathbf{G}, \mathbf{G}') &= \frac{1}{N^3} \int \sum_{\mathbf{G}\mathbf{G}'} e^{-i\mathbf{G}\cdot\mathbf{x}} e^{i\mathbf{G}'\cdot\mathbf{x}} d\tau = \frac{1}{N^3} \int \sum_{\mathbf{G}\mathbf{G}'} e^{i(\mathbf{G}'-\mathbf{G})\cdot\mathbf{x}} d\tau \\ &= \delta_{\mathbf{G}'\mathbf{G}} = \delta_{\mathbf{G}\mathbf{G}'}. \end{aligned} \tag{32}$$

With

$$\partial_m e^{i\mathbf{G}\cdot\mathbf{x}} = iG_m e^{i\mathbf{G}\cdot\mathbf{x}}, \tag{33}$$

the structure coefficients become

$$\begin{aligned} D(\mathbf{G}, \mathbf{G}', m) &= \int e^{-i\mathbf{G}\cdot\mathbf{x}} \; \partial_m \; e^{i\mathbf{G}'\cdot\mathbf{x}} d\tau \\ &= \int e^{-i\mathbf{G}\cdot\mathbf{x}} \; iG'_m e^{i\mathbf{G}'\cdot\mathbf{x}} d\tau \\ &= iG'_m \delta_{\mathbf{G}\mathbf{G}'}, \end{aligned} \tag{34}$$

where G'_m, m=1,2,3, is the m-th coordinate of $\mathbf{G}'$. In order to avoid confusion of indices, we replace the imaginary unit i with I. Eq. (29) then reads

$$\sum_{\mathbf{G};\; i=1}^{N} I \left(c_{ik_3} G_{k_2} - c_{ik_2} G_{k_3} \right) - \kappa \sum_{i=1}^{N} c_{ik_1} = 0. \tag{35}$$

These are $3N$ equations for determining the coefficients $c_{ik_{1,2,3}}$, representing an eigenvalue equation with eigenvalues κ and eigenvectors $c_{ik_{1,2,3}}$. The eigenvalue computation requires quadratic matrices, and therefore we obtain 3N eigenvalues and eigenvectors, numbered by the index j. Eq. (35) then becomes

$$\sum_{\mathbf{G};\; i=1}^{N} I \left(c_{ijk_3} G_{k_2} - c_{ijk_2} G_{k_3} \right) - \kappa_j \sum_{i=1}^{N} c_{ijk_1} = 0. \tag{36}$$

This equation has to be solved numerically.

The symmetries and boundary conditions of the definition space were specified by selecting suitable plane wave vectors G, and cartesian coordinates were used for simplicity. This adaptation of solid-state physics procedures would need to be modified for specific applications. For example, for spheres, spherical polar coordinates would have to be used, along with spherical harmonics as basis functions (also see [5]).

3 Summary

We have developed a numerical solution scheme for the Beltrami equation

$$\boldsymbol{\nabla} \times \mathbf{F} = \kappa \mathbf{F}. \tag{37}$$

This method was illustrated using plane waves as basis functions for a series expansion of F, and these basis functions had the form

$$\phi_{\mathbf{G}} = \frac{1}{\sqrt{N^3}} \sum_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{x}}. \tag{38}$$

The resulting eigenvalue equation then has to be solved numerically, for example, by using a numerical package like LAPACK, or a computer algebra system like Mathematica.

In this way, we are able to construct Beltrami functions of arbitrary form that could be used to describe parallel structures of vector fields, for example, to explain ball lightning or structures suitable for producing antigravity.

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