```
(%i1)  kill(all);
(%o0)  done

(%i1)  assume(r>0, L>0, m>0, M>0, G>0, c>0);
(%o1)  [r>0,L>0,m>0,M>0,G>0,c>0]
```

1 Solve S_r(r)

```
(%i2)  E19: E = (c^2*(diff(S_r(r),r)^2+L^2/r^2)+m^2*c^4)^(1/2)-m*M*G/r;
```

$$(E19)\quad E=\sqrt{c^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2+\frac{L^2}{r^2}\right)+c^4\,m^2}-\frac{G\,M\,m}{r}$$

```
(%i3)  Ea: expand(E19^2);
```

$$(Ea)\quad E^2=-\frac{2\,G\,M\,m\sqrt{c^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2+\frac{L^2\,c^2}{r^2}+c^4\,m^2}}{r}+c^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2+\frac{G^2\,M^2\,m^2}{r^2}+\frac{L^2\,c^2}{r^2}+c^4\,m^2$$

```
(%i4)  Eb: (Ea-(c^2*(diff(S_r(r),r,1))^2+(G^2*M^2*m^2)/r^2+
       (L^2*c^2)/r^2+c^4*m^2));
```

$$(Eb)\quad -c^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2-\frac{G^2\,M^2\,m^2}{r^2}-\frac{L^2\,c^2}{r^2}-c^4\,m^2+E^2=-\frac{2\,G\,M\,m\sqrt{c^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2+\frac{L^2\,c^2}{r^2}+c^4\,m^2}}{r}$$

```
(%i5)  Ec: Eb*r/(2*G*M*m);
```

$$(Ec)\quad \frac{r\left(-c^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2-\frac{G^2\,M^2\,m^2}{r^2}-\frac{L^2\,c^2}{r^2}-c^4\,m^2+E^2\right)}{2\,G\,M\,m}=-\sqrt{c^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2+\frac{L^2\,c^2}{r^2}+c^4\,m^2}$$

```
(%i6)  Ed: Ec^2*(4*G^2*M^2*m^2)/r^2;
```

$$(Ed)\quad \left(-c^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2-\frac{G^2\,M^2\,m^2}{r^2}-\frac{L^2\,c^2}{r^2}-c^4\,m^2+E^2\right)^2=\frac{4\,G^2\,M^2\,m^2\left(c^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2+\frac{L^2\,c^2}{r^2}+c^4\,m^2\right)}{r^2}$$

```
(%i7)  Ee: expand(Ed);
```

$$(Ee)\quad c^4\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^4+\frac{2\,G^2\,M^2\,c^2\,m^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2}{r^2}+\frac{2\,L^2\,c^4\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2}{r^2}+2\,c^6\,m^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2-2\,E^2\,c^2$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2+\frac{2\,G^2\,M^2\,c^4\,m^4}{r^2}+\frac{2\,L^2\,c^6\,m^2}{r^2}-\frac{2\,E^2\,G^2\,M^2\,m^2}{r^2}-\frac{2\,E^2\,L^2\,c^2}{r^2}+\frac{G^4\,M^4\,m^4}{r^4}+\frac{2\,G^2\,L^2\,M^2\,c^2\,m^2}{r^4}+\frac{L^4\,c^4}{r^4}$$
$$+c^8\,m^4-2\,E^2\,c^4\,m^2+E^4=\frac{4\,G^2\,M^2\,c^2\,m^2\left(\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)\right)^2}{r^2}+\frac{4\,G^2\,M^2\,c^4\,m^4}{r^2}+\frac{4\,G^2\,L^2\,M^2\,c^2\,m^2}{r^4}$$

```
(%i8)  Ef: radcan(solve(Ed, (diff(S_r(r),r))^1));
```

$$(Ef)\quad \left[\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)=-\frac{\sqrt{(E^2-c^4\,m^2)\,r^2+2\,E\,G\,M\,m\,r+G^2\,M^2\,m^2-L^2\,c^2}}{c\,r},\ \frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)=\right.$$
$$\frac{\sqrt{(E^2-c^4\,m^2)\,r^2+2\,E\,G\,M\,m\,r+G^2\,M^2\,m^2-L^2\,c^2}}{c\,r},\ \frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)=-\frac{\sqrt{(E^2-c^4\,m^2)\,r^2-2\,E\,G\,M\,m\,r+G^2\,M^2\,m^2-L^2\,c^2}}{c\,r},$$
$$\left.\frac{\mathrm{d}}{\mathrm{d}\,r}S_r(r)=\frac{\sqrt{(E^2-c^4\,m^2)\,r^2-2\,E\,G\,M\,m\,r+G^2\,M^2\,m^2-L^2\,c^2}}{c\,r}\right]$$

```
➔      /*ode2(Ef[1],S_r(r),r);*/;
```

2 Solve S_r(r) for a concrete parameter set

```
(%i62) str: [E=1000, L=200, m=1, M=10, G=1, c=20];
(str)  [E=1000,L=200,m=1,M=10,G=1,c=20]
```

(%i75)
```
str: [E=-7.6+400, L=2.215, m=1, M=10, G=1, c=20];
```

(str) $[E=392.4, L=2.215, m=1, M=10, G=1, c=20]$

(%i76)
```
Eg: ev(Ef, str);
```

(Eg) $$\left[\frac{d}{d\,r}S_r(r)=-\frac{\sqrt{-6022.24000000002\,r^2+7848.0\,r-1862.49}}{20\,r},\frac{d}{d\,r}S_r(r)=\frac{\sqrt{-6022.24000000002\,r^2+7848.0\,r-1862.49}}{20\,r},\frac{d}{d\,r}S_r(r)=-\frac{\sqrt{-6022.24000000002\,r^2-7848.0\,r-1862.49}}{20\,r},\frac{d}{d\,r}S_r(r)=\frac{\sqrt{-6022.24000000002\,r^2-7848.0\,r-1862.49}}{20\,r}\right]$$

```
(%i80)  Eh[1]: ode2(Eg[1],S_r(r),r);
        Eh[2]: ode2(Eg[2],S_r(r),r);
        Eh[3]: ode2(Eg[3],S_r(r),r);
        Eh[4]: ode2(Eg[1],S_r(r),r);
 rat: replaced -1862.49 by -186249/100 = -1862.49
rat: replaced 7848.0 by 7848/1 = 7848.0
rat: replaced -6022.24000000002 by -12171910020265/2021159904 = -6022.24000000002
rat: replaced -1862.49 by -186249/100 = -1862.49
rat: replaced 7848.0 by 7848/1 = 7848.0
rat: replaced -6022.24000000002 by -12171910020265/2021159904 = -6022.24000000002
```

(%o77) $$S_r(r)=\%c-(\sqrt{-304297750506625\,r^2+396551573164800\,r-94109752740024}-\frac{39655157316480\,\operatorname{asin}\left(\frac{396551573164800-608595501013250\,r}{60\sqrt{11862112761494933604527890}}\right)}{\sqrt{12171910020265}}+42\sqrt{53350199966}\,\operatorname{asin}\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right))/(400\sqrt{126322494})$$

```
rat: replaced -1862.49 by -186249/100 = -1862.49
rat: replaced 7848.0 by 7848/1 = 7848.0
rat: replaced -6022.24000000002 by -12171910020265/2021159904 = -6022.24000000002
rat: replaced -1862.49 by -186249/100 = -1862.49
rat: replaced 7848.0 by 7848/1 = 7848.0
rat: replaced -6022.24000000002 by -12171910020265/2021159904 = -6022.2400000000
2
```

(%o78) $$S_r(r)=(\sqrt{-304297750506625\,r^2+396551573164800\,r-94109752740024}-\frac{39655157316480\,\operatorname{asin}\left(\frac{396551573164800-608595501013250\,r}{60\sqrt{11862112761494933604527890}}\right)}{\sqrt{12171910020265}}+42\sqrt{53350199966}\,\operatorname{asin}\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right))/(400\sqrt{126322494})+\%c$$

```
rat: replaced -1862.49 by -186249/100 = -1862.49
rat: replaced -7848.0 by -7848/1 = -7848.0
rat: replaced -6022.24000000002 by -12171910020265/2021159904 = -6022.24000000002
rat: replaced -1862.49 by -186249/100 = -1862.49
rat: replaced -7848.0 by -7848/1 = -7848.0
rat: replaced -6022.24000000002 by -12171910020265/2021159904 = -6022.24000000002
```

(%o79) $$S_r(r)=\%c-(\%i\,(-42\sqrt{53350199966}\,\log\left(\frac{84\sqrt{53350199966}\sqrt{304297750506625\,r^2+396551573164800\,r+94109752740024}}{r}+\frac{188219505480048}{r}+396551573164800\right)+(39655157316480\log(10\sqrt{12171910020265}\sqrt{304297750506625\,r^2+396551573164800\,r+94109752740024}+608595501013250\,r+396551573164800))/\sqrt{12171910020265}+\sqrt{304297750506625\,r^2+396551573164800\,r+94109752740024}))/(400\sqrt{126322494})$$

```
rat: replaced -1862.49 by -186249/100 = -1862.49
rat: replaced 7848.0 by 7848/1 = 7848.0
rat: replaced -6022.24000000002 by -12171910020265/2021159904 = -6022.24000000002
rat: replaced -1862.49 by -186249/100 = -1862.49
rat: replaced 7848.0 by 7848/1 = 7848.0
rat: replaced -6022.24000000002 by -12171910020265/2021159904 = -6022.24000000002
```

(%o80) $$S_r(r)=\%c-(\sqrt{-304297750506625\,r^2+396551573164800\,r-94109752740024}-\frac{39655157316480\,\operatorname{asin}\left(\frac{396551573164800-608595501013250\,r}{60\sqrt{11862112761494933604527890}}\right)}{\sqrt{12171910020265}}+42\sqrt{53350199966}\,\operatorname{asin}\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right))/(400\sqrt{126322494})$$

```
(%i88)  Ei[1]: realpart(rhs(Eh[1]))-%c;
        Ei[2]: realpart(rhs(Eh[2]))-%c;
        Ei[3]: realpart(rhs(Eh[3]))-%c;
        Ei[4]: realpart(rhs(Eh[4]))-%c;
```

(%o85)

$$-\left(\cos\left(\frac{\operatorname{atan2}(0,-304297750506625\,r^2+396551573164800\,r-94109752740024)}{2}\right)\sqrt{\left|304297750506625\,r^2-396551573164800\,r+94109752740024\right|}-\left(39655157316480\,\operatorname{atan2}\left(\frac{396551573164800-608595501013250\,r}{60\sqrt{11862112761494933604527890}}+\sin\left(\frac{\operatorname{atan2}\left(0,1-\frac{(396551573164800-608595501013250\,r)^2}{427036059413817609763004040000}\right)}{2}\right)\sqrt{\left|\frac{(396551573164800-608595501013250\,r)^2}{427036059413817609763004040000}-1\right|},\cos\left(\frac{\operatorname{atan2}\left(0,1-\frac{(396551573164800-608595501013250\,r)^2}{427036059413817609763004040000}\right)}{2}\right)\sqrt{\left|\frac{(396551573164800-608595501013250\,r)^2}{427036059413817609763004040000}-1\right|}\right)\Big/\sqrt{12171910020265}+42\sqrt{53350199966}\,\operatorname{atan2}\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}+\sin\left(\frac{\operatorname{atan2}\left(0,1-\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right)^2\right)}{2}\right)\sqrt{\left|\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right)^2-1\right|}-\frac{6609192886080}{\sqrt{11862112761494933604527890}},\cos\left(\frac{\operatorname{atan2}\left(0,1-\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right)^2\right)}{2}\right)\sqrt{\left|\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right)^2-1\right|}\right)\Big/\left(400\sqrt{126322494}\right)$$

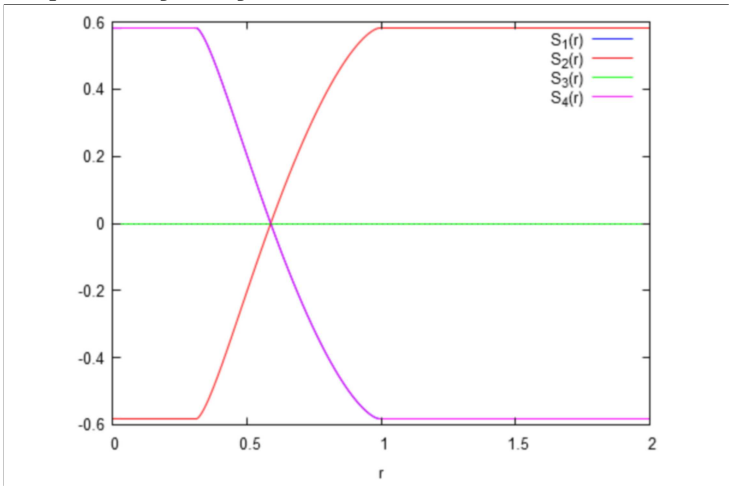
(%o86)

$$\left(\cos\left(\frac{\operatorname{atan2}(0,-304297750506625\,r^2+396551573164800\,r-94109752740024)}{2}\right)\sqrt{\left|304297750506625\,r^2-396551573164800\,r+94109752740024\right|}-\left(39655157316480\,\operatorname{atan2}\left(\frac{396551573164800-608595501013250\,r}{60\sqrt{11862112761494933604527890}}+\sin\left(\frac{\operatorname{atan2}\left(0,1-\frac{(396551573164800-608595501013250\,r)^2}{427036059413817609763004040000}\right)}{2}\right)\sqrt{\left|\frac{(396551573164800-608595501013250\,r)^2}{427036059413817609763004040000}-1\right|},\cos\left(\frac{\operatorname{atan2}\left(0,1-\frac{(396551573164800-608595501013250\,r)^2}{427036059413817609763004040000}\right)}{2}\right)\sqrt{\left|\frac{(396551573164800-608595501013250\,r)^2}{427036059413817609763004040000}-1\right|}\right)\Big/\sqrt{12171910020265}+42\sqrt{53350199966}\,\operatorname{atan2}\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}+\sin\left(\frac{\operatorname{atan2}\left(0,1-\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right)^2\right)}{2}\right)\sqrt{\left|\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right)^2-1\right|}-\frac{6609192886080}{\sqrt{11862112761494933604527890}},\cos\left(\frac{\operatorname{atan2}\left(0,1-\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right)^2\right)}{2}\right)\sqrt{\left|\left(\frac{15684958790004}{5\sqrt{11862112761494933604527890}\,r}-\frac{6609192886080}{\sqrt{11862112761494933604527890}}\right)^2-1\right|}\right)\Big/\left(400\sqrt{126322494}\right)$$

```
(%i94)  wxplot2d([Ei[1],Ei[2],Ei[3],Ei[4]], [r,0,2],
        [legend, "S_1(r)", "S_2(r)", "S_3(r)", "S_4(r)"])$
```

```
 plot2d: expression evaluates to non-n
umeric value somewhere in plotting range.
plot2d: expression evaluates to non-numeric value some
where in plotting range.
plot2d: expression evaluates to non-numeric value so
mewhere in plotting range.
```

(%t94)




```
(%i95)  plot2d([Ei[1],Ei[2],Ei[3],Ei[4]], [r,0,2],
        [legend, "S_1(r)", "S_2(r)", "S_3(r)", "S_4(r)"],
        [gnuplot_term, "png linewidth 2 font 'Arial' 16 size 800,600"],
        [gnuplot_out_file, "D:/Doc/Artikel-Eck/ECE-Theorie/Paper426/FigS.png"]
        )$
```

```
 plot2d: expression evaluates to non-numeric value
 somewhere in plotting range.
plot2d: expression evaluates to non-numeric valu
e somewhere in plotting range.
plot2d: expression evaluates to non-numeric value somewhere in pl
otting range.
```