```
(%i1)  kill(all);
(%o0)  done
```

1 Equations of central motion in coordinates (r,phi)

```
(%i1)  depends([r,phi],t);
(%o1)  [r(t),φ(t)]
```

```
(%i2)  diff(r,t,2) = diff(phi,t)^2*r - M*G/r^2;
```

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\,r=\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,\varphi\right)^2 r-\frac{G\,M}{r^2}\tag{\%o2}$$

```
(%i3)  diff(phi,t,2) = -2*diff(phi,t)*diff(r,t)/r;
```

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\,\varphi=-\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,\varphi\right)\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,r\right)}{r}\tag{\%o3}$$

2 Non-relativistic Hamilton equations I p^2 = p_r^2 + p_phi^2

```
(%i4)  H: (p_r^2+p_phi^2)/(2*m)-m*M*G/q_r;
```

$$\frac{{p_r}^2+{p_{phi}}^2}{2\,m}-\frac{G\,M\,m}{q_r}\tag{H}$$

2.1 First Hamilton equations

```
(%i5)  H1: q_rd = diff(H,p_r);
```

$$q_{rd}=\frac{p_r}{m}\tag{H1}$$

```
(%i6)  H2: q_phid = diff(H,p_phi);
```

$$q_{phid}=\frac{p_{phi}}{m}\tag{H2}$$

2.2 Second Hamilton equations

```
(%i7)  H3: p_rd = -diff(H,q_r);
```

$$p_{rd}=-\frac{G\,M\,m}{{q_r}^2}\tag{H3}$$

```
(%i8)  H4: p_phid = -diff(H,q_phi);
```

$$p_{phid}=0\tag{H4}$$

3 Non-relativistic Hamilton equations II p^2 = p_r^2 + p_phi^2 / q_r^2

```
(%i9)  H: (p_r^2+1/q_r^2*p_phi^2)/(2*m)-m*M*G/q_r;
```

$$\frac{\frac{{p_{phi}}^2}{{q_r}^2}+{p_r}^2}{2\,m}-\frac{G\,M\,m}{q_r}\tag{H}$$

3.1 First Hamilton equations

```
(%i10)  H1: q_rd = diff(H,p_r);
```

$$q_{rd}=\frac{p_r}{m}\tag{H1}$$

```
(%i11)  H2: q_phid = diff(H,p_phi);
```

$$q_{phid}=\frac{p_{phi}}{m\,{q_r}^2}\tag{H2}$$

3.2 Second Hamilton equations

```
(%i12)  H3: p_rd = -diff(H,q_r);
```

(H3) $$p_{rd}=\frac{p_{phi}^2}{m\,q_r^3}-\frac{G\,M\,m}{q_r^2}$$

```
(%i13)  H4: p_phid = -diff(H,q_phi);
```

(H4) $$p_{phid}=0$$

4 Relativistic Hamilton equations I gamma defined by velocities p_r^2 + p_phi^2

```
(%i14)  gamma: (1-(q_rd^2+q_phid^2)/c^2)^(-1/2);
```

(gamma) $$\frac{1}{\sqrt{1-\frac{q_{rd}^2+q_{phid}^2}{c^2}}}$$

```
(%i15)  H: 1/gamma*((p_r^2+p_phi^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;
```

(H) $$\left(\frac{p_r^2+p_{phi}^2}{m}+c^2\,m\right)\sqrt{1-\frac{q_{rd}^2+q_{phid}^2}{c^2}}-\frac{G\,M\,m}{q_r}$$

4.1 First Hamilton equations

```
(%i16)  H1: q_rd = diff(H,p_r);
```

(H1) $$q_{rd}=\frac{2\,p_r\sqrt{1-\frac{q_{rd}^2+q_{phid}^2}{c^2}}}{m}$$

```
(%i17)  H2: q_phid = diff(H,p_phi);
```

(H2) $$q_{phid}=\frac{2\,p_{phi}\sqrt{1-\frac{q_{rd}^2+q_{phid}^2}{c^2}}}{m}$$

4.2 Second Hamilton equations

```
(%i18)  H3: p_rd = (-diff(H,q_r));
```

(H3) $$p_{rd}=-\frac{G\,M\,m}{q_r^2}$$

```
(%i19)  H4: p_phid = -diff(H,q_phi);
```

(H4) $$p_{phid}=0$$

4.3 Re-insert gamma

```
(%i23)  ratsubst(%gamma, gamma, H1);
        ratsubst(%gamma, gamma, H2);
        expand(ratsubst(%gamma, gamma, H3));
        ratsubst(%gamma, gamma, H4);
```

(%o20) $$q_{rd}=\frac{2\,p_r}{\gamma\,m}$$

(%o21) $$q_{phid}=\frac{2\,p_{phi}}{\gamma\,m}$$

(%o22) $$p_{rd}=-\frac{G\,M\,m}{q_r^2}$$

(%o23) $$p_{phid}=0$$

5 Relativistic Hamilton equations I gamma defined by velocities q_r^2 + q_phi^2

(%i24) `gamma: (1-(q_rd^2+q_phid^2)/c^2)^(-1/2);`

(gamma)
$$\frac{1}{\sqrt{1-\frac{q_{rd}^2+q_{phid}^2}{c^2}}}$$

(%i25) `H: 1/gamma*((p_r^2+p_phi^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;`

(H)
$$\left(\frac{p_r^2+p_{phi}^2}{m}+c^2\ m\right)\sqrt{1-\frac{q_{rd}^2+q_{phid}^2}{c^2}}-\frac{G\ M\ m}{q_r}$$

5.1 First Hamilton equations

(%i26) `H1: q_rd = diff(H,p_r);`

(H1)
$$q_{rd}=\frac{2\ p_r\sqrt{1-\frac{q_{rd}^2+q_{phid}^2}{c^2}}}{m}$$

(%i27) `H2: q_phid = diff(H,p_phi);`

(H2)
$$q_{phid}=\frac{2\ p_{phi}\sqrt{1-\frac{q_{rd}^2+q_{phid}^2}{c^2}}}{m}$$

5.2 Second Hamilton equations

(%i28) `H3: p_rd = (-diff(H,q_r));`

(H3)
$$p_{rd}=-\frac{G\ M\ m}{q_r^2}$$

(%i29) `H4: p_phid = -diff(H,q_phi);`

(H4)
$$p_{phid}=0$$

5.3 Re-insert gamma

(%i33)
```
ratsubst(%gamma, gamma, H1);
ratsubst(%gamma, gamma, H2);
expand(ratsubst(%gamma, gamma, H3));
ratsubst(%gamma, gamma, H4);
```

(%o30)
$$q_{rd}=\frac{2\ p_r}{\gamma\ m}$$

(%o31)
$$q_{phid}=\frac{2\ p_{phi}}{\gamma\ m}$$

(%o32)
$$p_{rd}=-\frac{G\ M\ m}{q_r^2}$$

(%o33)
$$p_{phid}=0$$

6 Relativistic Hamilton equations II gamma defined by velocities p_r^2 + q_r^2*p_phi^2

(%i34) `gamma: (1-(q_rd^2+q_r^2*q_phid^2)/c^2)^(-1/2);`

(gamma)
$$\frac{1}{\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\ q_r^2}{c^2}}}$$

(%i35) `H: 1/gamma*((p_r^2+p_phi^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;`

(H)
$$\left(\frac{p_r^2+p_{phi}^2}{m}+c^2\ m\right)\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\ q_r^2}{c^2}}-\frac{G\ M\ m}{q_r}$$

6.1 First Hamilton equations

(%i36)
```
H1: q_rd = diff(H,p_r);
```

(H1)
$$q_{rd}=\frac{2\,p_r\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}{m}$$

(%i37)
```
H2: q_phid = diff(H,p_phi);
```

(H2)
$$q_{phid}=\frac{2\,p_{phi}\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}{m}$$

6.2 Second Hamilton equations

(%i38)
```
H3: p_rd = (-diff(H,q_r));
```

(H3)
$$p_{rd}=\frac{\left(\frac{p_r^2+p_{phi}^2}{m}+c^2\,m\right)q_{phid}^2\,q_r}{c^2\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}-\frac{G\,M\,m}{q_r^2}$$

(%i39)
```
H4: p_phid = -diff(H,q_phi);
```

(H4)
$$p_{phid}=0$$

6.3 Re-insert gamma

(%i43)
```
ratsubst(%gamma, gamma, H1);
ratsubst(%gamma, gamma, H2);
expand(ratsubst(%gamma, gamma, H3));
ratsubst(%gamma, gamma, H4);
```

(%o40)
$$q_{rd}=\frac{2\,p_r}{\gamma\,m}$$

(%o41)
$$q_{phid}=\frac{2\,p_{phi}}{\gamma\,m}$$

(%o42)
$$p_{rd}=\frac{\gamma\,p_r^2\,q_{phid}^2\,q_r}{c^2\,m}+\frac{\gamma\,p_{phi}^2\,q_{phid}^2\,q_r}{c^2\,m}+\gamma\,m\,q_{phid}^2\,q_r-\frac{G\,M\,m}{q_r^2}$$

(%o43)
$$p_{phid}=0$$

7 Relativistic Hamilton equations III

(%i44)
```
gamma: (1-(p_r^2+p_phi^2)/(m^2*c^2))^(-1/2);
```

(gamma)
$$\frac{1}{\sqrt{1-\frac{p_r^2+p_{phi}^2}{c^2\,m^2}}}$$

(%i45)
```
H: 1/gamma*((p_r^2+p_phi^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;
```

(H)
$$\sqrt{1-\frac{p_r^2+p_{phi}^2}{c^2\,m^2}}\left(\frac{p_r^2+p_{phi}^2}{m}+c^2\,m\right)-\frac{G\,M\,m}{q_r}$$

7.1 First Hamilton equations

(%i46)
```
H1: q_rd = diff(H,p_r);
```

(H1)
$$q_{rd}=\frac{2\,p_r\sqrt{1-\frac{p_r^2+p_{phi}^2}{c^2\,m^2}}}{m}-\frac{p_r\left(\frac{p_r^2+p_{phi}^2}{m}+c^2\,m\right)}{c^2\,m^2\sqrt{1-\frac{p_r^2+p_{phi}^2}{c^2\,m^2}}}$$

(%i47)
```
H2: q_phid = diff(H,p_phi);
```

(H2)
$$q_{phid}=\frac{2\,p_{phi}\sqrt{1-\frac{p_r{}^2+p_{phi}{}^2}{c^2\,m^2}}}{m}-\frac{p_{phi}\left(\frac{p_r{}^2+p_{phi}{}^2}{m}+c^2\,m\right)}{c^2\,m^2\sqrt{1-\frac{p_r{}^2+p_{phi}{}^2}{c^2\,m^2}}}$$

7.2 Second Hamilton equations

(%i48)
```
H3: p_rd = (-diff(H,q_r));
```

(H3)
$$p_{rd}=-\frac{G\,M\,m}{q_r{}^2}$$

(%i49)
```
H4: p_phid = -diff(H,q_phi);
```

(H4)
$$p_{phid}=0$$

7.3 Re-insert gamma

(%i53)
```
ratsubst(%gamma, gamma, H1);
ratsubst(%gamma, gamma, H2);
expand(ratsubst(%gamma, gamma, H3));
ratsubst(%gamma, gamma, H4);
```

(%o50)
$$q_{rd}=-\frac{\gamma^2\,p_r{}^3+\left(\gamma^2\,p_{phi}{}^2+\left(\gamma^2-2\right)\,c^2\,m^2\right)\,p_r}{\gamma\,c^2\,m^3}$$

(%o51)
$$q_{phid}=-\frac{\gamma^2\,p_{phi}\,p_r{}^2+\gamma^2\,p_{phi}{}^3+\left(\gamma^2-2\right)\,c^2\,m^2\,p_{phi}}{\gamma\,c^2\,m^3}$$

(%o52)
$$p_{rd}=-\frac{G\,M\,m}{q_r{}^2}$$

(%o53)
$$p_{phid}=0$$

8 Relativistic Hamilton equations IV

(%i54)
```
gamma: (1-(p_r^2+1/q_r^2*p_phi^2)/(m^2*c^2))^(-1/2);
```

(gamma)
$$\frac{1}{\sqrt{1-\frac{\frac{p_{phi}{}^2}{q_r{}^2}+p_r{}^2}{c^2\,m^2}}}$$

(%i55)
```
H: 1/gamma*((p_r^2+p_phi^2/q_r^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;
```

(H)
$$\sqrt{1-\frac{\frac{p_{phi}{}^2}{q_r{}^2}+p_r{}^2}{c^2\,m^2}}\left(\frac{\frac{p_{phi}{}^2}{q_r{}^2}+p_r{}^2}{m}+c^2\,m\right)-\frac{G\,M\,m}{q_r}$$

8.1 First Hamilton equations

(%i56)
```
H1: q_rd = diff(H,p_r);
```

(H1)
$$q_{rd}=\frac{2\,p_r\sqrt{1-\frac{\frac{p_{phi}{}^2}{q_r{}^2}+p_r{}^2}{c^2\,m^2}}}{m}-\frac{p_r\left(\frac{\frac{p_{phi}{}^2}{q_r{}^2}+p_r{}^2}{m}+c^2\,m\right)}{c^2\,m^2\sqrt{1-\frac{\frac{p_{phi}{}^2}{q_r{}^2}+p_r{}^2}{c^2\,m^2}}}$$

(%i57)
```
H2: q_phid = diff(H,p_phi);
```

(H2)
$$q_{phid}=\frac{2\,p_{phi}\sqrt{1-\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{c^2\,m^2}}}{m\,q_r^2}-\frac{p_{phi}\left(\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{m}+c^2\,m\right)}{c^2\,m^2\sqrt{1-\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{c^2\,m^2}}\,q_r^2}$$

8.2 Second Hamilton equations

(%i58)
```
H3: p_rd = (-diff(H,q_r));
```

(H3)
$$p_{rd}=-\frac{G\,M\,m}{q_r^2}-\frac{p_{phi}^2\left(\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{m}+c^2\,m\right)}{c^2\,m^2\sqrt{1-\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{c^2\,m^2}}\,q_r^3}+\frac{2\,p_{phi}^2\sqrt{1-\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{c^2\,m^2}}}{m\,q_r^3}$$

(%i59)
```
H4: p_phid = -diff(H,q_phi);
```

(H4)
$$p_{phid}=0$$

8.3 Re-insert gamma

(%i63)
```
ratsubst(%gamma, gamma, H1);
ratsubst(%gamma, gamma, H2);
expand(ratsubst(%gamma, gamma, H3));
ratsubst(%gamma, gamma, H4);
```

(%o60)
$$q_{rd}=-\frac{(\gamma^2\,p_r^3+(\gamma^2-2)\,c^2\,m^2\,p_r)\,q_r^2+\gamma^2\,p_{phi}^2\,p_r}{\gamma\,c^2\,m^3\,q_r^2}$$

(%o61)
$$q_{phid}=-\frac{(\gamma^2\,p_{phi}\,p_r^2+(\gamma^2-2)\,c^2\,m^2\,p_{phi})\,q_r^2+\gamma^2\,p_{phi}^3}{\gamma\,c^2\,m^3\,q_r^4}$$

(%o62)
$$p_{rd}=-\frac{G\,M\,m}{q_r^2}-\frac{\gamma\,p_{phi}^2\,p_r^2}{c^2\,m^3\,q_r^3}-\frac{\gamma\,p_{phi}^2}{m\,q_r^3}+\frac{2\,p_{phi}^2}{\gamma\,m\,q_r^3}-\frac{\gamma\,p_{phi}^4}{c^2\,m^3\,q_r^5}$$

(%o63)
$$p_{phid}=0$$