586 (5): Unique Solution for ω_x, ω_y and ω_z

This solution considers the antisymmetric equations together with the assumption:

$$\underline{\nabla} \times \underline{A} = -\underline{\omega} \times \underline{A} \quad -(1)$$

so

$$\underline{B} = 2\underline{\nabla} \times \underline{A} \quad -(2)$$

and

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(3)$$

From the Ampère Law: $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad -(4)$

it follows that:

$$\underline{A} = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3x' \quad -(5)$$

as in Jackson, chapter 5.

The antisymmetric law:

$$\frac{\partial A_z}{\partial y} + \frac{\partial A_y}{\partial Z} = \omega_y A_z + \omega_z A_y \quad -(6)$$

$$\frac{\partial A_x}{\partial Z} + \frac{\partial A_z}{\partial X} = \omega_z A_x + \omega_x A_z \quad -(7)$$

$$\frac{\partial A_y}{\partial X} + \frac{\partial A_x}{\partial y} = \omega_x A_y + \omega_y A_x \quad -(8)$$

is solved simultaneously with eq. (1), which is:

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial Z} = -(\omega_y A_z - \omega_z A_y) \quad -(9)$$

$$\frac{\partial A_x}{\partial Z} - \frac{\partial A_z}{\partial X} = -(\omega_z A_x - \omega_x A_z) \quad -(10)$$

$$\frac{\partial A_y}{\partial X} - \frac{\partial A_x}{\partial y} = -(\omega_x A_y - \omega_y A_x) \quad -(11)$$

Add (6) and (9):

$$\frac{\partial A_z}{\partial y} = \omega_z A_y \quad - (12)$$

Add (7) and (10):

$$\frac{\partial A_x}{\partial z} = \omega_x A_z \quad - (13)$$

Add (8) and (11):

$$\frac{\partial A_y}{\partial x} = \omega_y A_x \quad - (14)$$

Subtract (9) from (6):

$$\frac{\partial A_y}{\partial z} = \omega_y A_z \quad - (15)$$

Subtract (10) from (7)

$$\frac{\partial A_z}{\partial x} = \omega_z A_x \quad - (16)$$

Subtract (11) from (8):

$$\frac{\partial A_x}{\partial y} = \omega_x A_y \quad - (17)$$

Add (12) and (16):

$$\left(\frac{\partial A_z}{\partial y} + \frac{\partial A_z}{\partial x} \right) = \omega_z \left(A_y + A_x \right) \quad - (18)$$

Add (13) and (17):

$$\left(\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial y} \right) = \omega_x \left(A_z + A_y \right) \quad - (19)$$

Add (14) and (15):

$$\left(\frac{\partial A_y}{\partial x} + \frac{\partial A_y}{\partial z} \right) = \omega_y \left(A_x + A_z \right) \quad - (20)$$

Eqs. (18) to (20) give ω_x, ω_y and ω_z

) given the assumption (1).

Now use eq. (5) for a circular current loop in the limit:

$$r \gg a \qquad -(21)$$

In spherical polar coordinates:

$$A_\phi = \frac{\mu_0 I a^2}{4r^2} \sin\theta \qquad -(22)$$

so

$$\underline{A} = A_\phi \, \underline{e}_\phi \qquad -(23)$$

As in Note 386(3), this translates into:

$$\underline{A} = \frac{\mu_0 I a^2}{4(X^2+Y^2+Z^2)^{3/2}} \left(-Y\underline{i} + X\underline{j} \right) \qquad -(24)$$

This potential is used with eqs. (18) to (20) to find ω_X, ω_Y and ω_Z, Q.E.D.

In general, the integral (5) is evaluated as in Jackson, page 182, 3rd edition for a circular loop:

$$A_\phi(r,\theta) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi' \, d\phi'}{(a^2 + r^2 - 2ar\sin\theta\cos\phi')^{1/2}} \qquad -(25)$$

where a is the radius of the current loop and r is the distance from the centre of the loop to a point P (Jackson's Figure (5.5)).

Eq. (25) is a well known problem that gives the solution:

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi}\frac{4Ia}{(a^2+r^2+2ar\sin\theta)^{1/2}}\left(\frac{(2-k^2)K(k)-2E(k)}{k^2}\right) \quad -(26)$$

here K and E are the complete elliptic integrals with argument:

$$k^2 = \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta} \quad -(27)$$

The components of the magnetic flux density in spherical polar coordinates are:

$$B_r = \frac{2}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi \sin\theta) \quad -(28)$$

$$B_\theta = -\frac{2}{r}\frac{\partial}{\partial r}(rA_\phi) \quad -(29)$$

$$B_\phi = 0 \quad -(30)$$

i.e.

$$\underline{B} = 2\,\underline{\nabla}\times\underline{A} \quad -(31)$$

To translate into Cartesian components:

$$r^2 = x^2 + y^2 + z^2 \quad -(32)$$

$$\sin\theta = \left(1-\frac{z^2}{r^2}\right)^{1/2} \quad -(33)$$

The spin connection can be computed from eqs. (21), (37), (18), (19) and (20). In the limit:

$a \gg r$; $a \ll r$; or $\theta \ll 1$:

$$A_\phi(r,\theta) = \frac{\mu_0 I a^2 r \sin\theta}{4(a^2+r^2)^{3/2}}\left(1 + \frac{15 a^2 r^2 \sin^2\theta}{8(a^2+r^2)^2} + \dots\right) \quad - (34)$$

and:

$$B_r = \frac{2\mu_0 I a^2 \cos\theta}{2(a^2+r^2)^{3/2}}\left(1 + \frac{15 a^2 r^2 \sin^2\theta}{4(a^2+r^2)^2} + \dots\right) \quad - (35)$$

$$B_\theta = \frac{-2\mu_0 I a^2 \sin\theta}{4(a^2+r^2)^{5/2}}\left(2a^2 - r^2 + \frac{15 a^2 r^2 \sin^2\theta\,(4a^2-3r^2)}{8(a^2+r^2)^2} + \dots\right) \quad - (36)$$

1) Near the axis, $\theta \ll 1$ $\quad - (37)$

2) Near the centre of the loop:

$r \ll a$ $\quad - (38)$

3) Far from the loop:

$r \gg a$ $\quad - (39)$

Far from the loop:

$$B_r = \frac{2\mu_0}{2\pi}\left(I\pi a^2\right)\frac{\cos\theta}{r^3} \quad - (40)$$

$$B_\theta = \frac{2\mu_0}{4\pi}\left(I\pi a^2\right)\frac{\sin\theta}{r^3} \quad - (41)$$

and

$$A_\phi = \frac{\mu_0 I a^2}{4 r^2}\sin\theta \quad - (42)$$

) These are the dipole fields and potential. The factor 2 appears in the numerator of eqs. (40) and (41) because of eqs. (1) and (2).

So the spin connections can be worked out for this problem, giving a complete solution that conserves antisymmetry, Q.E.D.