374(3): Basics of the Lagrangian Method

These are given in any good textbook, for example chapter 5 of Marion and Thornton, "Classical Dynamics". The method is based on the Hamiltonian Principle of Least Action:

$$\delta \int (T - U) \, dt = 0 \quad -(1)$$

where

$$T = T(\dot{x}_i), \quad U = U(x_i) \quad -(2)$$

The Lagrangian is defined by:

$$\mathcal{L} = T(\dot{x}_i) - U(x_i) \quad -(3)$$

where T is the kinetic energy and U is the potential energy. In orbital theory:

$$T = T(\dot{\underline{r}}), \quad U = U(\underline{r}) \quad -(4)$$

so the Lagrangian is well defined:

$$\mathcal{L} = T(\dot{\underline{r}}) - U(\underline{r}) \quad -(5)$$

In consequence there is only one Lagrange variable, the position vector $\underline{r}$.

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2m} \dot{\underline{r}} \cdot \dot{\underline{r}} + \frac{mMG}{r} \quad -(6)$$

and the Euler Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(7)$$

As in note 374(3) eqs. (6) and (7) lead to:

$$\underline{F} = \dot{\underline{p}} = \frac{\partial \mathcal{L}}{\partial \underline{r}} = m\ddot{\underline{r}} \quad -(8)$$

2) Eq. (8) is valid for any momentum $\underline{p}$, and can be derived entirely without the use of the Lagrangian method. For plane polar coordinates:

$$\underline{r} = r\,\underline{e}_r \quad -(9)$$

and

$$\underline{v} = \dot{\underline{r}} = \frac{d}{dt}\left(r\,\underline{e}_r\right) = \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \quad -(10)$$

$$= \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \quad -(11)$$

The acceleration is:

$$\underline{a} = \dot{\underline{v}} = \ddot{\underline{r}} = \left(\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left(\dot{r}\dot{\phi} + r\ddot{\phi}\right)\underline{e}_\phi$$

Therefore eq. (8) is:

$$\underline{F} = m\underline{a} = m\left(\left(\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left(\dot{r}\dot{\phi} + r\ddot{\phi}\right)\underline{e}_\phi\right)$$

$$= -\frac{mMG}{r^2}\,\underline{e}_r \quad -(12)$$

It follows that:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{MG}{r^2} \quad -(13)$$

and

$$\dot{r}\dot{\phi} + r\ddot{\phi} = 0 \quad -(14)$$

The angular momentum is defined as constant, for planar orbital motion, and is:

$$\underline{L} = \underline{r} \times \underline{p} \quad -(15)$$

From eqs. (9) and (11):

$$\underline{L} = mr^2 \dot{\phi} \, \underline{k} \quad -(16)$$

so

$$\dot{\phi} = \frac{L}{mr^2} \quad -(17)$$

Eqs. (13), (14) and (17) give:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(18)$$

and this result can be derived numerically by solving eqs. (13), (14) and (17) to give:

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} \quad -(19)$$

and

$$r = \int \frac{dr}{d\phi} \, d\phi \quad -(20)$$

This checks the numerical method against the well known analytical result (18).

Here:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(21)$$

$$\epsilon^2 = 1 + \frac{2HL^2}{m^3 M^2 G^2} \quad -(22)$$

The hamiltonian H in eq. (22) is defined as

$$H = \frac{1}{2} m \, \underline{\dot{r}} \cdot \underline{\dot{r}} - \frac{mMG}{r} \quad -(23)$$

$$= \frac{1}{2} mv^2 - \frac{mMG}{r}$$

4) In Eq. (23):

$$v^2 = mG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(24)$$

where

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(25)$$

is the semi major axis of the ellipse (18). Here α is the half right latitude and ϵ is the eccentricity.

From eqs. (23) and (24):

$$H = \frac{1}{2} m M G\left(\frac{2}{r} - \frac{1}{a}\right) - \frac{mMG}{r}$$

$$= -\frac{mMG}{2a} \quad -(26)$$

These well known results can be obtained from a numerical solution of the simultaneous differential equations:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{MG}{r^2} \quad -(27)$$

$$\dot{r}\dot{\phi} + r\ddot{\phi} = 0 \quad -(28)$$

$$\dot{\phi} = \frac{L}{mr^2} \quad -(29)$$

This should be done as a baseline computation.

For fluid dynamics (UFT 363), the momentum is:

$$\underline{p} = m\underline{v} = m(\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi) \quad -(30)$$

5) where

$$x = 1 + \frac{\partial R_r}{\partial r} \quad -(31)$$

Therefore:

$$\dot{\underline{r}} = x\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi \quad -(32)$$

and the Lagrangian is defined by eqs. (6) and (32). The momentum $\underline{p}$ is correctly defined by the canonical equation (3):

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(33)$$

This is one of the Lagrange equations of motion. The acceleration is defined by:

$$\ddot{\underline{r}} = \frac{d\dot{\underline{r}}}{dt} = \frac{d}{dt}\left(x\dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi\right). \quad -(33)$$

In eq. (31), R_r is the position of a fluid element in fluid dynamics, the spacetime being considered as a fluid:

$$R_r = R_r\left(r(t), \phi(t), t\right). \quad -(34)$$

Therefore in general, R_r is a function of r, ϕ and t. Assuming for the sake of simplicity that R_r is time independent then the acceleration is calculated with:

$$\ddot{\underline{r}} = x\ddot{r}\,\underline{e}_r + x\dot{r}\,\dot{\underline{e}}_r + \dot{r}\dot{\phi}\,\underline{e}_\phi + r\ddot{\phi}\,\underline{e}_\phi + r\dot{\phi}\,\dot{\underline{e}}_\phi \quad -(35)$$

$$= x\ddot{r}\,\underline{e}_r + x\dot{r}\dot{\phi}\,\underline{e}_\phi + \dot{r}\dot{\phi}\,\underline{e}_\phi + r\ddot{\phi}\,\underline{e}_\phi - r\dot{\phi}^2\,\underline{e}_r$$

$$= \left(x\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + (x+1)\,\dot{r}\dot{\phi}\,\underline{e}_\phi + r\ddot{\phi}\,\underline{e}_\phi$$

6) Therefore from eq. (8):

$$\underline{F} = m\left(\left(x\ddot{r} - r\dot{\phi}^2\right)\underline{e}_r + \left((x+1)\dot{r}\dot{\phi} + r\ddot{\phi}\right)\underline{e}_\phi\right)$$

$$= -\frac{mMG}{r^2}\underline{e}_r \quad -(36)$$

This gives the simultaneous equations:

$$x\ddot{r} - r\dot{\phi}^2 = -\frac{MG}{r^2} \quad -(37)$$

$$(x+1)\dot{r}\dot{\phi} + r\ddot{\phi} = 0 \quad -(38)$$

The angular momentum is:

$$\underline{L} = \underline{r} \times \underline{p} \quad -(39)$$

$$= m\begin{vmatrix} \underline{e}_r & \underline{e}_\phi & \underline{k} \\ r & 0 & 0 \\ x\dot{r} & r\dot{\phi} & 0 \end{vmatrix}$$

$$= mr^2\dot{\phi}$$

so

$$\dot{\phi} = \frac{L}{mr^2} \quad -(40)$$

where L is defined as a constant of motion.

So eqs. (37), (38) and (40) can be solved numerically to give:

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} \quad -(41)$$

7) and to give the orbit:

$$r = \int \frac{dr}{d\phi} d\phi \quad -(42)$$

In order to determine $\partial R_r / \partial r$, other equations of the fluid dynamics are needed, as in Note 374 (1). In the first instance a model of $\partial R_r / \partial r$ can be used, and the orbit (42) computed in terms of this model function. In an incompressible fluid there is the additional equation:

$$\underline{\nabla} \cdot \underline{v} = \underline{\nabla} \cdot \left(x \dot{r} \, \underline{e}_r + r \dot{\phi} \, \underline{e}_\phi \right) = 0 \quad -(43)$$

This gives:

$$\frac{1}{r} \left(x \dot{r} + \dot{\phi} \frac{\partial r}{\partial \phi} \right) + \frac{\partial}{\partial r} \left(x \dot{r} \right) + \frac{\partial \dot{\phi}}{\partial \phi} \quad -(44)$$

Using eqs. (37), (38), (40) and (44), the orbit and x can both be determined.

Caveat

From eqs. (6) and (30), the lagrangian is

$$\mathcal{L} = \frac{1}{2} m \left(x^2 \dot{r}^2 + r^2 \dot{\phi}^2 \right) + \frac{mMG}{r} \quad -(45)$$

However, if it is assumed that the proper Lagrange variables are r and ϕ, the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(46)$$

gives
$$x^2 \ddot{r} = r\dot{\phi}^2 - \frac{mG}{r^2} \quad -(47)$$

and this is not the same as the correct eq. (37).

The use of Eq. (46) is incorrect because the lagrangian (45) does not satisfy the requirement (3) derived from the Hamilton Principle of Least Action, eq. (1). The lagrangian (45) contains a kinetic energy:

$$T = \frac{1}{2} m \left(x^2 \dot{r}^2 + r^2 \dot{\phi}^2 \right) \quad -(48)$$

$$= T\left(x^2, \dot{r}, \dot{\phi}\right)$$

The required format of the kinetic energy is:

$$T = T\left(\dot{r}, \dot{\phi}\right). \quad -(49)$$